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THE GAME OF SIM: A WINNING STRATEGY FOR THE SECOND PLAYER

ERNEST MEAD, ALEXANDER ROSA, CHARLOTTE HUANG,
McMaster University, Hamilton, Ontario

The game of SIM [**2**, **4**] is played by two players on the six vertices of the complete graph K_6: the first and second player color alternately the 15 edges of K_6 with black and red respectively. A player wins if he forces his opponent to complete a monochromatic triangle. It is well known [3] that the Ramsey number $r(3,3)$ equals 6 (i.e., six is the smallest integer n such that in any coloring of the edges of the complete graph K_n on n vertices by two colors, there must be a monochromatic triangle); therefore a tie is impossible. It is also well known (see, e.g., [**1**]) that then one of the players must possess a winning strategy. It turns out that it is the second player who can be assured of his win. We were able to devise a strategy for the second player and to show that it is a winning strategy; however, a simpler (in terms of the rules to be followed) winning strategy is still desirable.

A few basic observations. The first player (i.e., the player who makes the first move) uses black and the second player uses red. The vertex-set and the edge-set of a graph G are denoted by $V(G)$ and $E(G)$ respectively; it will always be clear from the content whether G is undirected or directed. The elements of $E(K_6)$ are (unordered) pairs (i,j), $i \neq j$. We assume that in any game played, a player will never make a move that completes a triangle of his own color unless he is forced to do so. In the latter case, the game stops immediately before the losing move is made.

A *position* of a particular game after k moves is a subgraph of K_6 with k edges, each of which is colored black or red. Evidently, we must have $k \leqq 14$; let us admit also $k = 0$ and call the corresponding position *initial*. On the other hand, a position is *terminal* if the player who has to make the next move, must complete a monochromatic triangle (and therefore loses). A terminal position is said to be a *black terminal* or a *red terminal* according to whether the loser is the second player or the first player, respectively.

One could conceivably determine the winning strategy for one of the players in the game of SIM as follows:

Form a directed graph G whose vertices are all possible positions of all possible games, and whose edges are the moves, i.e., $(A,B) \in E(G)$ if there exists a particular game in which position B arises from position A after a single move. In the graph G just described, one looks for a set $S \subseteq V(G)$ with the properties:

(1) For every vertex $X \in V(G)$, $X \notin S$ such that $(Y, X) \in E(G)$ for some $Y \in S$, there is a vertex $T \in S$ such that $(X, T) \in E(G)$.

(2) $Y,\ T \in S \Rightarrow (Y, T) \notin E(G)$.

(3) Either the initial position or the position following the initial position is in S, and S includes some of the terminal positions.

The set S is a kernel of a certain subgraph of G, namely of the subgraph spanned by the vertices of S and all the successors of the vertices of S in G. Since S can obviously contain terminal positions of only one color, it determines a winning strategy for the player with the color of the terminal vertices in S.

However, although this consideration will prove useful, it probably cannot be used for actual determination of the winning strategy in the game of SIM. The reason for this is that to construct the graph G means, in effect, to perform a complete analysis of the game. Although it is evidently sufficient to take as vertices of the graph G the isomorphism classes of positions (considered as graphs), the number of possibilities (and of essentially different games of SIM which can be played) is probably large enough to prevent anyone from determining a winning strategy by using the described approach.

Our approach. However, we do not abandon the described approach altogether. We proceed as follows: we devise a set R_0 of heuristic rules to be followed by the second player. Now if one forms a graph G_0 in the same way as one formed the graph G except that the vertices of G_0 are all possible positions of all possible games in which the second player *follows the rules of* R_0, then obviously G_0 is a subgraph of G. Let S_0 be the set of vertices of G_0 which consists of all positions obtained after moves of the second player. Now one has to check whether S_0 has the properties (1), (2), (3); that is verify whether for every vertex X representing a position after a move of the first player, there is a vertex T in S_0 that can be obtained from X by a single move of the second player. If yes, then S_0 determines a winning strategy for the second player. If not, then one modifies the set of rules R_0 to obtain a new set of rules R_1; then one forms the graph G_1 and the set of vertices S_1, etc. One hopes that eventually one will arrive at a set of rules $R_i = R^*$ such that the set of vertices S_i of the corresponding graph G_i will have properties (1), (2), (3). Obviously, whether this will occur depends on how one changes the set of rules R_j to obtain R_{j+1}. One feels that R_{j+1} has to be, in a certain sense, a "refinement" of R_j, so as to take care of possibilities not covered by R_j.

Of course, this is a very loose description of a procedure which has been used by us and which eventually led to a determination of a winning strategy R^* for the second player. In the next section we describe in a more precise manner this winning strategy.

Let us call the edges of the complement of any position *free edges*. Suppose a, b, c are three vertices of K_6 such that precisely one of the three edges (a, b), (a, c), (b, c) is colored red and the remaining two edges are free. Then coloring one

one of those two edges red is said to *create a loser* (since then the third edge cannot be colored red and it becomes a losing move for the second player). If precisely one of the edges is colored black and the other two are free, then coloring either of these edges red is said to *create a partial mixed triangle.* If two of the three edges are colored (not both red) and the third edge is free, then coloring that edge red is said to constitute a *completion of a mixed triangle.*

We started with three heuristic rules considered in a hierarchy which can be formulated as follows: the second player should consider as possible moves only those free edges which when colored red will not complete a mixed triangle with two black edges (that is, unless he has no other choice) and from among these he should color red in his next move that one which, when becoming red, will

(1) create a minimum possible number of losers,

(2) complete a maximum possible number of mixed triangles,

(3) create a maximum possible number of partial mixed triangles.

Here the hierarchy of the rules is to be understood in the following sense: first one uses the rule (1) to single out those free edges which satisfy it, and only then the rule (2) is used to distinguish between free edges satisfying (1), etc. One could consider these three rules as a strategy R_0 mentioned above. Although this strategy cannot guarantee a win for the second player, it turned out to be a reasonably good approximation of what proved to be a winning strategy for the second player. We now skip the intermediate stages of the procedure described at the beginning of this section, and proceed to a description of the winning strategy.

Elements of the winning strategy. First of all, there is a need for a finer partition of $E(K_6)$ than the one given by

$$E(K_6) = A_i \cup B_i \cup N_i$$

where A_i, B_i and N_i, respectively, are the sets of black, red and free edges, respectively, after the ith move of a particular fixed game ($i = 0, 1, \cdots, t;\ t \leqq 14$).

A free edge (a, b) is said to be a *loser* for the first player (the second player, respectively) if there is a vertex v (different from a, b) such that both edges (a, v), (b, v) are colored black (red, respectively).

Now, we shall partition the set N_i as follows:

$$N_i = C_i \cup D_i \cup E_i \cup F_i$$

(some of the sets C_i, D_i, E_i, F_i may be empty), where the elements of C_i are losers for the first player, the elements of D_i are losers for the second player, the elements of E_i are losers for both players and the elements of F_i are the remaining free edges, called neutral edges.

Thus after the ith move, $i = 0, 1, \cdots, t,\ t \leqq 14$, the set $E(K_6)$ is partitioned by

$$E(K_6) = A_i \cup B_i \cup C_i \cup D_i \cup E_i \cup F_i$$

(some of the sets may be empty; for instance, $F_0 = E(K_6)$, $A_0 = B_0 = C_0 = D_0 = E_0 = \varnothing$).

When playing the game it is convenient to denote a loser for the first player by a dotted-red line with the implication that this edge can be colored red by the second player whenever he wishes. Similarly it is convenient to denote the losers for the second player and the losers for both players by dotted-black and dotted-red-black lines, respectively. Henceforth assume that all such lines are drawn as soon as the first player has completed a move and before the second player starts the decision process for his move. Also note that, unless specified otherwise, a line described as red can be solid or dotted-red and that a line described as black can be solid, dotted-black or dotted-red-black.

Assume that the second player is considering the edge (a, c) for a move (where (a, c) can be a neutral or dotted-red edge). If, with respect to any other vertex b, the two edges (a, b) and (b, c) are such that

(a) Both are red, then we say that he will *ruin a safe move* if he colors (a, c). (If one of the edges (a, b), (b, c) is dotted-red and the base of another red triangle, then we say that he will ruin a *hypothetically* safe move, otherwise a *valid* safe move.)

(b) One is red and the other neutral, then we say that he will *create a loser* (a *hypothetical* loser if the red edge is dotted, otherwise a *valid* loser).

(c) Both are colored (but not both red), then we say that he will *complete a mixed triangle.*

(d) One is black (including dotted-red-black) and one is neutral, then we say that he will *create a partial mixed triangle.*

A winning strategy for the second player is:

Rule 1. For the second move (i.e., when answering the first move of the first player), the second player should color red an edge which has no common *vertex* with the edge chosen by the first player.

Rule 2. For any move other than the second when there is at least one neutral edge, consider only these neutral edges and apply the following rules in a hierarchy in the same sense as described in the previous section:

(1) Ruin a minimum number of valid safe moves.
(2) Create a minimum number of losers (valid and hypothetical).
(3) Ruin a minimum number of hypothetically safe moves.
(4) Complete a maximum number of mixed triangles.
(5) Create a maximum number of partial mixed triangles.
(6) Create a minimum number of valid losers.

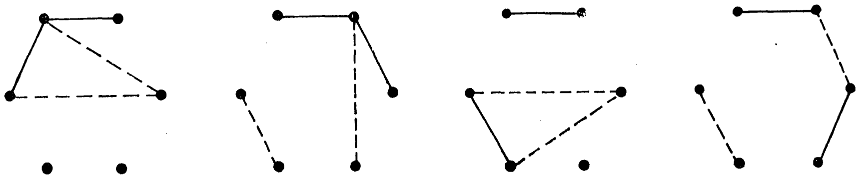
Then color any one of the edges satisfying the above rules.

Rule 3. For any move other than the second when there are no neutral edges, consider only the dotted-red edges and apply the same steps as in Rule 2.

Thus at each stage in the play of the game the free edges each act as the base of four triangles. It is a comparison of the colorings of these four triangles for each edge that determines which edge should be colored. Although the steps used in applying the strategy may seem numerous, with very little practice they are easy to use when the game is actually being played.

It turns out that the edge to be chosen as the next move by the second player is determined by the rule 2 uniquely up to an isomorphism. Notice that the second move (i.e., the first play to be made by the second player, rule 1) is in fact an exception denying the rule 2! If the second player would make that move by rule 2 and continue with rules 2 and 3, he could lose! We do not have the slightest idea why it is so.

Unfortunately, we cannot present a short and/or elegant proof of the fact that our strategy is a winning one. Most likely, in view of the complexity of the strategy, such a proof does not exist. We performed our proof in an exhaustive manner by playing all possible games in which the second player uses our strategy, in two ways: by hand, and by using a computer. When making the verification by hand, we were able to make substantial reductions by elimination of isomorphic positions. When making the verification on a computer (a CDC6400), we did not provide for a reduction of isomorphic positions, so that in several cases isomorphic games were played by the computer. For the third move (which is made by the first player) there are four essentially different possibilities, each one followed, of course, by a unique fourth move made by the second player. These first four completed moves were included into the input data and thus the program was run four times, each with different input data (see Figure 1 and Table 1).



Case 1 Case 2 Case 3 Case 4

———————— black - - - - - - - - - - - red

FIG. 1

TABLE 1

Case	Computer time (in sec)	Number of games played
1	73.820	509
2	75.812	644
3	80.918	569
4	96.626	742
Total	327.176	2464

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POLYA'S ENUMERATION FORMULA BY EXAMPLE

ALAN TUCKER, SUNY at Stony Brook

1. Introduction. One of the most important results in combinatorial mathematics is Polya's enumeration formula. This formula constructs a generating function for the number of different ways to mark the corners of an unoriented figure using a given set of labels. For example, if the figure were an unoriented cube and one could color the corners black or white, Polya's formula would yield the following generating function, called the *pattern inventory*:

$$b^8 + b^7w + 3b^6w^2 + 3b^5w^3 + 7b^4w^4 + 3b^3w^5 + 3b^2w^6 + bw^7 + w^8,$$

where the coefficient of b^iw^j is the number of distinct colorings with i black corners and j white corners. More general nongeometric applications are also possible. Polya's work was motivated by some problems of isomer enumeration in organic chemistry (half a century earlier, Kukulé had confirmed the structure of the benzene ring by comparing the observed number of isomers of $C_6H_3R_3$ against the theoretical number of isomers in competing hypothetical structures).

As a result of the recent surge of student interest in computer science and operations research on campuses, most universities now offer an upper-level undergraduate course in combinatorial mathematics. Indeed, the combinatorics course at Stony Brook has become very popular (this year its enrollment was the largest of all upper-level undergraduate mathematics courses), in large part, because it heavily emphasizes problem-solving and minimizes theory. In this spirit, the course develops Polya's formula in an "experimental" fashion with several sets of small problems which lead the students to infer the formula's structure.

Fifteen years ago, there was no American textbook which presented Polya's formula. Recent combinatorics texts which include the formula all present it in full generality. Unfortunately, the general proof of the formula is complicated and not intuitive. Indeed, when Professor Polya taught part of the combinatorial mathematics course at Stanford (for which this author was a teaching assistant), he omitted a proof of his formula and gave only examples. A cookbook use of the formula is all that is needed for most applications. On the other hand, the derivation of the formula has much to recommend it: the basis of the formula is Burnside's theorem whose proof is a marriage of elementary group theory and a standard combinatorial argument; and it makes the most elegant and powerful use of generating functions to be found anywhere in basic combinatorial mathematics. After several unrewarding attempts at a standard presentation of Polya's formula, this author evolved a presentation of a slightly simplified form of the formula that retains the pedagogical aspects of the standard approach but replaces the usual complexities with a score of simple, suggestive exercises.

2. Basic concepts. We wish to find a formula for the pattern inventory of the number of ways to color the corners (or sides or faces) of an unoriented geometric figure choosing from n colors. We develop our theory around the sample problem of coloring corners of an unoriented (floating in three dimensions) square with black or white. By observation, we see that the pattern inventory for this problem is $b^4 + b^3w + 2b^2w^2 + bw^3 + w^4$ (the computations would be unduly messy if a less obvious sample problem were chosen). Since the inventory polynomial cannot be factored, we cannot sneak up upon a mathematical formulation by working backward. Thus we must start from scratch; but what is there to work with? The most obvious thing is the set of 16 colorings of the fixed square (see Figure 1). As a forerunner of the pattern inventory formula of the unoriented square, we observe that $(b + w)^4 = b^4 + 4b^3w + 6b^2w^2 + 4bw^3 + w^4$ is the pattern inventory of the 2-colorings of the fixed square (each $(b + w)$ term corresponds to the color choices at a given corner). We can partition these 16 colorings into subsets of colorings that are all equivalent when the square is unoriented, i.e., partition into equivalence classes. As yet, the underlying equivalence relation has no mathematical foundation. Note, as a measure of the difficulty of our whole problem, that the equivalence classes vary greatly in size.

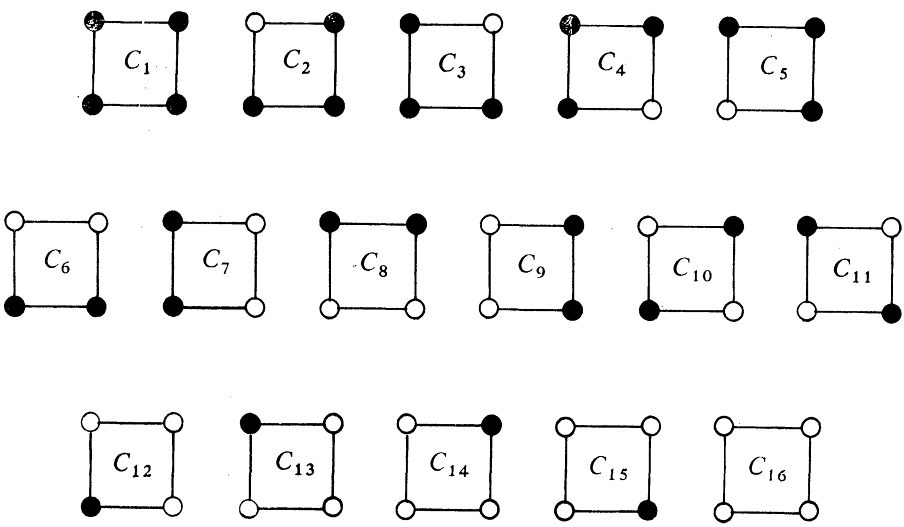

FIG. 1. Colorings of the Fixed Square

For information about this equivalence, we turn our attention to the motions π that map the square into itself, called the *symmetries* of the square (see Figure 2). Clearly, two colorings are equivalent when there exists a motion which acts to transform one into the other. The problem of finding all symmetries is often difficult, but with the square they can be enumerated as the ways of moving corner a to any other corner and simultaneously placing corner b on the clockwise or counterclockwise side of a. For any regular n-gon, it turns out that the motions are the set of all possible rotations and reflections (in Figure 2, π_1, π_2, π_3, π_4 are rotations of 0°, 90°, 180°, 270°, respectively, π_5, π_6, reflections about opposite sides, and π_7, π_8, reflections

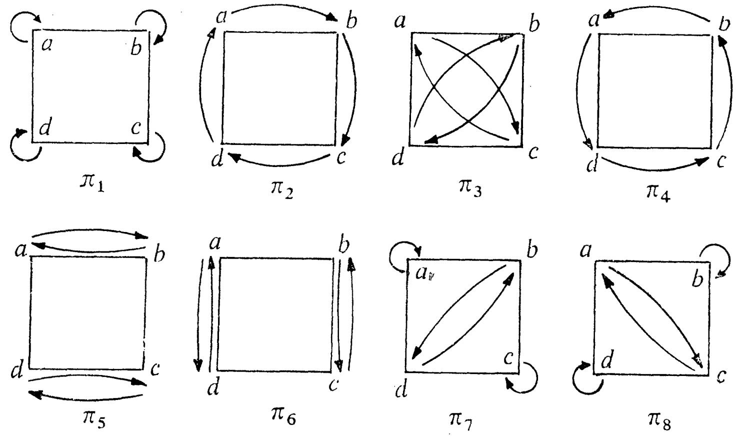

FIG. 2

Symmetries of the Square; the use of arrows indicating the mapping at each corner makes it easier for the student to see the fundamental role which the cyclic decomposition will later play.

about opposite corners). The motions are naturally characterized by the way they permute the corners. Thus the motion π_3 can be defined as the corner permutation $(ac)(bd)$ (the representation of permutations as a product of disjoint cycles is central to our development). In permuting the corners, the motions induce permutations of the colorings of the corners. We let π_i^* denote the permutation of the 16 2-colorings induced by the motion π_i. Thus colorings C and C' are equivalent if there exists a motion π_i such that $\pi_i^*(C) = C'$.

It is well known that the symmetries of a geometric figure form a group under the associative operation of composition. Furthermore, it is exactly the properties of a group that induce the equivalence relation on the colorings, i.e., closure in the group makes the relation transitive, existence of an identity makes it reflexive, and existence of inverses makes it symmetric. (This correspondence is no coincidence—historically, the concept of a group grew out of the study of various symmetries and their induced equivalences.)

Now we formalize the relation between π_i and π_i^*. We say a group G *acts on* a set S if each element $\pi \in G$ induces a permutation π_i^* on S, called the *action* of π_i, and if $(\pi_i \circ \pi_j)^* = \pi_i^* \circ \pi_j^*$. The second condition implies that the set of π_i^*'s also forms a group, but the π_i's are easier to work with than the π_i^*'s; for example in our current problem, a permutation on 4 elements is preferable to a permutation on 16 elements. Note that two different π's could have the same action—such a situation arises naturally when dealing with subsets of the 2-colorings; for example, all motions have the same action on the subset $\{C_1, C_{16}\}$. The following summarizes our discussion:

THEOREM 1. *Suppose the group G (symmetries of the square) acts on a set S (2-colorings of the fixed square). Then this action induces a partition of S into equivalence classes with the equivalence relation defined by $C \sim C' \Leftrightarrow$ there exists $\pi \in G$ whose action takes C to C'* (i.e., $\pi^*(C) = C'$).

Although the concept of the action of a group seems to portend much abstract mathematics, we use it only in the intuitive setting of symmetries of a square acting on colorings of the square. It will eliminate the need for always having a separate group of π^*'s.

3. Burnside's Lemma. Instead of seeking a formula for the pattern inventory for the different 2-colorings of the unoriented square, let us simplify the problem and ask for a formula for just the total number of different 2-colorings, i.e., the size of the pattern inventory. The following modified form of Burnside's Lemma answers this question in the generality of Theorem 1.

THEOREM 2 (Burnside, 1897). *The number N of equivalence classes in the equivalence relation on S (the 2-colorings) induced by the action on S of a group G (symmetries of the square) is given by the formula:*

$$(*)\qquad N = \sum_{\pi \in G} \psi(\pi)$$

where $\psi(\pi)$ is the number of elements of S left fixed by π^*, the action of π.

Although in general, $\psi(\pi)$ is only indirectly a function of π, in our square coloring problem, this dependence is quite obvious. Since the result of the theorem is basically combinatorial, it is best to extract all the algebra in the proof and place it in a separate lemma.

LEMMA 1. *Let G be a group which acts on a set S. For $a \in S$, let $\eta(a)$ be the number of π's whose action fixes a (i.e., $\pi^*(a) = a$) and let S_a be the equivalence class containing a in the equivalence relation on S induced by G. Then $\sum_{b \in S_a} \eta(b) = |G|$.*

Proof. Let $T(a,b)$ be the set of distinct π's such that $\pi^*(a) = b$. Let $T(a,a) = \pi_1, \pi_2, \cdots, \pi_k$ and let $\pi_x(a) = b$. Then for $1 \leqq i \leqq k$, $\pi_i \circ \pi_x \in T(a,b)$; moreover $\pi_i \circ \pi_x = \pi_j \circ \pi_x \Rightarrow \pi_i \circ \pi_x \circ \pi_x^{-1} = \pi_j \circ \pi_x \circ \pi_x^{-1} \Rightarrow \pi_i = \pi_j$, i.e., $i = j$. Suppose $\pi_y \in T(a,b)$. Then $\pi_y \circ \pi_x^{-1} \in T(a,a)$ and so for some j, $\pi_j = \pi_y \circ \pi_x^{-1}$; then $\pi_j \circ \pi_x = \pi_y \circ \pi_x^{-1} \circ \pi_x = \pi_y$. Thus $T(a,b) = \{\pi_1 \circ \pi_x, \cdots, \pi_k \circ \pi_x\}$. We can also show that $T(b,a) = \{\pi^{-1} : \pi \in T(a,b)\}$. Thus $\eta(a) = |T(a,a)| = |T(a,b)| = |T(b,a)| = |T(b,b)| = \eta(b)$. Since the sets $T(a,b)$, $b \in S_a$, exhaust G, $\sum_{b \in S_a} \eta(b) = \sum_{b \in S_a} |T(a,b)| = |G|$. (QED)

Note that we additionally have proved that $|T(a,a)| \cdot |S_a| = |G|$. Moreover, $T(a,a)$ is a subgroup and the $T(a,b)$, $b \in S_a$, are its cosets. Thus the equation $|T(a,a)| \cdot |S_a| = |G|$ is a special case of Lagrange's theorem for subgroups. Now we complete the proof of Theorem 2.

Proof of Theorem 2. By Lemma 1, $\sum_{b \in S_a} \eta(b) = |G|$. Then

$$\sum_{\text{all } c \in S} \eta(c) = \sum_{\substack{\text{each} \\ \text{eq class } S_a}} \left(\sum_{b \in S_a} \eta(b) \right) = N \cdot |G|; \text{ or } N = \frac{1}{|G|} \sum_{c \in S} \eta(c).$$

It remains to show that $\sum_{c \in S} \eta(c) = \sum_{\pi \in G} \psi(\pi)$. Define

$$\gamma(a, \pi) = \begin{cases} 1, & \pi^*(a) = a \\ 0, & \pi^*(a) \neq a \end{cases}.$$

So $\eta(c) = \sum_{\pi \in G} \gamma(c, \pi)$ and $\psi(\pi) = \sum_{c \in S} \gamma(c, \pi)$. Thus $\sum_{c \in S} \eta(c) = \sum_{c \in S} \sum_{\pi \in G} \gamma(c, \pi) = \sum_{\pi \in G} \sum_{c \in S} \gamma(c, \pi) = \sum_{\pi \in G} \psi(\pi)$. Now (*) follows. In words, $\sum_{c \in S} \eta(c)$ and $\sum_{\pi \in G} \psi(\pi)$ both count up all instances of some c fixed by the action of some π: the first expression sums over the c's and the second over the π's. (QED)

Several examples would now be in order. A typical application involves counting the number of different necklaces consisting of a circular string of 5 beads that can be formed by beads of 3 colors (S is the set of all 3^5 strings with beads in fixed positions and G is the group of different rotations of the string). At this point, the necklace question would be very difficult if the string had 6 beads (the problem is in determining the $\psi(\pi_i)$ — if the size is not a prime number, it is extremely difficult to count directly the number of strings left fixed by some π_i^*).

4. Discovering the cycle index. Without further theory, we would find it very difficult to apply formula (*) to most coloring problems. When the set S in Theorem 2 is large, such as 3-colorings of a cube, it is impractical to determine the coloring permutations π_i^*, and hence to determine the $\psi(\pi)$, explicitly. We shall show that $\psi(\pi)$ can be calculated directly from the simpler π (thus justifying the notation $\psi(\pi)$). This approach leads to greatly shortened computations. The steps leading to the simplified calculation of $\psi(\pi)$ can be motivated by many mini-examples which allow the student to anticipate the final formula.

Let us apply formula (*) to the 2-colorings of the square. We shall determine the number of 2-colorings left fixed by π_i empirically. (For simplicity we shall write "left fixed by π_i" in place of "left fixed by the action of π_i" — most students naturally think of a coloring being left fixed by a motion, rather than by its induced action.) As we determine $\psi(\pi)$ for successive motions π_i in Figure 2, we look for a pattern that would enable us mathematically to predict which colorings (and implicitly, how many colorings) will be left fixed by the π_i. It is helpful to make a table of the π_i's and the colorings that they leave fixed; see columns i and ii in Figure 3 (columns iii and iv are used later). It will become evident as we collect this data that the number of colorings left fixed by π_i can be determined from the cyclic decomposition of π_i. The 0° rotation π_1 leaves each corner fixed and hence it leaves each coloring fixed. Next consider the 90° motion π_2 which cyclicly permutes corners a, b, c, d (throughout the following discussion, it is assumed that for each motion the instructor asks the class which colorings are left fixed and why — and only afterwards gives his own explanation; this discussion can become dull and tedious for students if they are not involved in the development). In terms of corners, a coloring is left fixed if each corner has the same color after the motion as it did before. Since π_2 takes a to b, then a coloring left fixed by π_2 must have the same color at a as at b. Similarly, such a coloring must have the same color at b as at c, the same color at c as at d, and the same at d as at a. Taken together these conditions imply that only the colorings of all white or all black corners, C_1 and C_{16}, are left fixed. In general, a coloring C will be left fixed by π_i if and only if for each corner s, the color at s is the same as the color at $\pi(s)$ (thus keeping the color at $\pi(s)$ unchanged). Next we consider the 180° motion π_3. Looking at the depiction of π_3 in Figure 2, we see that π_3 causes corners a and c to interchange and corners b and d to interchange. It follows that a coloring left fixed by the action of π_3 must have the same color at corners a and c and the same color at b and d (no further conditions are needed). With two color choices for a, c and with two color choices for b, d, we can construct $2 \cdot 2 = 4$ colorings which will be left fixed, namely C_1, C_{10}, C_{11}, C_{16}. The motion π_4 is similar to π_2 and only C_1 and C_{16} are left fixed. The horizontal flip π_5 interchanges a and b and interchanges c and d. Again we must have two pairs of like-colored corners when constructing the colorings that will be left fixed by π_5. Like π_3, the motion π_5 will leave $2 \cdot 2 = 4$ colorings fixed.

(iv) Inventory of Colorings Left Fixed by π_i^*	(i) Motion π_i	(ii) Colorings Left Fixed by π_i^*	(iii) Cycle Structure Represen-tation
$(b+w)^4 = b^4+4b^3w+6b^2w^2+4bw^3+w^4$	π_1	16: all colorings	x_1^4
$(b^4+w^4)^1 = b^4 + w^4$	π_2	2: C_1, C_{16}	x_4^1
$(b^2+w^2)^2 = b^4 + 2b^2w^2 + w^4$	π_3	4: $C_1, C_{10}, C_{11}, C_{16}$	x_2^2
$(b^4+w^4)^1 = b^4 + w^4$	π_4	2: C_1, C_{16}	x_4^1
$(b^2+w^2)^2 = b^4 + 2b^2w^2 + w^4$	π_5	4: C_1, C_6, C_8, C_{16}	x_2^2
$(b^2+w^2)^2 = b^4 + 2b^2w^2 + w^4$	π_6	4: C_1, C_7, C_9, C_{16}	x_2^2
$(b+w)^2(b^2+w^2) = b^4+2b^3w+2b^2w^2+2bw^3+w^4$	π_7	8: C_1, C_2, C_4, C_{10}, $C_{11}, C_{13}, C_{15}, C_{16}$	$x_1^2x_2^1$
$(b+w)^2(b^2+w^2) = b^4+2b^3w+2b^2w^2+2bw^3+w^4$	π_8	8: C_1, C_3, C_5, C_{10}, $C_{11}, C_{12}, C_{14}, C_{16}$	$x_1^2x_2^1$
Total $= 8b^4+8b^3w+16b^2w^2+8bw^3+8w^4$		$P_G = \frac{1}{8}(x_1^4+2x_4^1+3x_2^2+2x_1^2x_2^1)$	

FIG. 3

A pattern is becoming clear. The student should now be able quickly to predict that π_6 will also leave $2 \cdot 2 = 4$ colorings fixed, for again the motion interchanges two pairs of corners. Formally, an interchange is a cyclic permutation on 2 elements. All our enumeration of fixed colorings has been based on the fact that if π_i cyclicly permutes a subset of corners (that is, the corners form a cycle of π_i), then those corners must be the same color in any coloring left fixed. Thus all we need to do is get a disjoint-cycle representation of a π_i and use the number of cycles. For future use, let us also classify the cycles by their length. It will prove convenient to encode a motion's cycle information in an expression of the form

$$x_1^{n_1} \cdot x_2^{n_2} \cdots x_r^{n_r},$$

where n_i is the number of i-cycles, cycles of length i. This expression is called the *cycle structure representation* of a motion. Observe that $n_1 + 2n_2 + \cdots kn_k = \left|\begin{matrix}\text{number of}\\ \text{corners}\end{matrix}\right|$. Now we add column iii to Figure 3 in which we write the cycle structure representation of each motion. So for π_2 and π_4 we enter x_4^1 and for π_3, π_5 and π_6 we enter x_2^2, but what about π_1? Previously, it sufficed to say that π_1 leaves all colorings fixed. Now it is time to point out that an element left fixed by a permutation is classified as a 1-cycle. Thus π_1 is really four 1-cycles. Its cycle structure representation is x_1^4. Since all the corners in each cycle must be one of the two colors in a fixed coloring, we predict *a posteriori* that π_1 leaves $2^4 = 16$ colorings fixed. For any π_i, the number of colorings left fixed will be given by setting each x_j equal to 2 (or, in general, the number of colors available) in the cycle structure representation of π_i. Finally we turn to π_7 and π_8. For each motion, the cycle structure representation is seen to be $x_1^2 \cdot x_2^1$ and thus for each we can find $2^2 \cdot 2 = 8$ colorings that are left fixed.

According to Theorem 2, we sum the numbers in column ii of Figure 3 and divide by 8 to obtain the number of 2-colorings of the unoriented square. There is a simpler way: algebraically sum the cycle structure representations of each motion, collecting like terms, and then divide by 8. From column iii, we obtain

$$\frac{1}{8}\left(x_1^4 + 2x_4^1 + 3x_2^2 + 2x_1^2x_2^1\right).$$

This expression is called the *cycle index* $P_G(x_1, x_2, \cdots, x_k)$ for our group G of motions. By setting each $x_i = 2$ in P_G, i.e., $P_G(2, 2, \cdots, 2)$, we get the same answer as before (before, the steps were reversed: we set $x_i = 2$ in each cycle structure representation and then added). Another advantage to the latter approach is that for any m, $P_G(m, m, \cdots, m)$ is the number of m-colorings of an unoriented square. The argument used to derive this coloring counting formula with the cycle index is valid for colorings of any set with associated symmetries (as an example, we can use the cycle index formula to rework a necklace counting problem for any number of colors). We shall state this as a corollary of Theorem 2. We emphasize again that although we were looking at properties of the π_i^*'s, the actions of π_i's, the new formula involves only the π_i's.

COROLLARY 1. *Let T be a set of elements and G be a group of permutations of T which acts to induce an equivalence relation on the colorings of T. Then the number of nonequivalent m-colorings of T is given by $P_G(m, m, \cdots, m)$.*

5. Discovering Polya's formula. We are now ready to return to our original goal of a formula for the pattern inventory. The pattern inventory can be considered as giving the results of several formula (*)-type counting subproblems. In the case of 2-colorings of the unoriented square, we divide the colorings in Figure 1 into sets based on the number of black and white corners: $S_0 = \{C_1\}$, $S_1 = \{C_2, C_3, C_4, C_5\}$ $S_2 = \{C_6, C_7, C_8, C_9, C_{10}, C_{11}\}$, $S_3 = \{C_{12}, C_{13}, C_{14}, C_{15}\}$ and $S_4 = \{C_{16}\}$. In the pattern inventory, the coefficient of b^3w is the number of different colorings with three blacks. This is the result obtained by (*) for the counting problem where G, the set of motions of the square, acts on the set S_1. The coefficient of b^4 is the result when S_0 is the set. In general, the coefficient of $b^{4-i}w^i$ is the result of (*) when S_i is the set on which G acts. (One must check that G does indeed act on each S_i, i.e., show each motion induces an action (permutation) on S_i; as noted before, different π's may induce the same action on S_i.) Let us try to solve these five subproblems simultaneously. That is, we will list in a row the numbers of 2-colorings left fixed in each subproblem by the action of π_1, then below this row we will list the numbers left fixed in each subproblem by π_2, then by π_3, etc. (see column iv in Figure 7); then we total up the first column (the first number in each row), and divide by 8, total up the second column and divide by 8, etc. Since the action of π_1 leaves all C's fixed, the first row is 1, 4, 6, 4, 1. Let us put this data in the same form as the pattern inventory. We write: $b^4 + 4b^3w + 6b^2w^2 + 4bw^3 + w^4$; this is an inventory of fixed colorings. Note that to get the pattern inventory we do not have to add columns separately; we can simply add up all these inventories, collect terms, and divide by 8. For π_1, the inventory of fixed colorings is an inventory of all colorings. As observed at the beginning, this inventory is simply $(b + w)^4 = (b + w)(b + w)(b + w)(b + w)$, one $(b + w)$ for each corner. For π_2, the inventory is $b^4 + w^4$ (or suggestively $(b^4 + w^4)$). For π_3, we find by observation that the inventory is $b^4 + 2b^2w^2 + w^4$. This expression factors into $(b^2 + w^2)^2$. Just as we did before when counting the total number of colorings fixed by the action of some π, let us look for a "pattern" in the inventories of fixed colorings. Again the key to the "pattern" is the fact that in a coloring fixed by π, all corners in a cycle of π must have the same color. Since π_2 has one cycle involving all four corners, the possibilities are thus all corners black or all corners white; hence the inventory is $b^4 + w^4$. The motion π_3 has two 2-cycles (ac) and (bd). Each 2-cycle uses two blacks or two whites in a fixed coloring; hence the inventory of a cycle of size two is $b^2 + w^2$. The possibilities with two such cycles have the inventory $(b^2 + w^2)(b^2 + w^2)$. Thus the inventory of fixed colorings for π_i is a product of factors $(b^j + w^j)$, one factor for each cycle of π_i with j equal to the size of the cycle. So for each π_i, we need to know the number of cycles in π_i of each length. But this is exactly the information encoded in the cycle structure representation. Indeed, setting $x_j = (b^j + w^j)$ in the representation yields precisely the inventory of fixed colorings for π_i. By this method we compute the rest of the inventories of fixed colorings. For π_7 especially, the inventory should be checked against the list of colorings in column ii. As noted above, the pattern inventory is obtained by adding together the inventories of fixed colorings, collecting like-power terms and dividing by 8. In turn, the inventories are obtained by setting $x_j = (b^j + w^j)$ in each cycle structure representation and expanding the resulting expressions. As before in section 4, we get a more compact formula and save some computations by first adding together the cycle structure representations, then setting each $x_j = (b^j + w^j)$, and doing the polynomial algebra all at once. Again the first step in this approach (followed by division by 8) yields the cycle index $P_G(x_1, x_2, \cdots, x_k)$. Thus by setting $x_j = (b^j + w^j)$ in P_G, i.e., $P_G((b + w), (b^2 + w^2), \cdots, (b^k + w^k))$, we obtain the pattern inventory.

If three colors, black, white and green, were permitted, each cycle of size j would have an inventory of $(b^j + w^j + g^j)$ in a fixed coloring; so we would set $x_j = (b^j + w^j + g^j)$ in P_G. The preceding argument applies for any number of colors and any figure. In greater generality we have the following theorem (note again that only π_i's are involved in our formula):

THEOREM 3 (Polya's Enumeration Formula). *Let T be a set of elements and G be a group of permutations of T which acts to induce an equivalence relation on the colorings of T. The inventory of nonequivalent colorings on T using colors $c_1, c_2, \cdots, c_m$ is given by the generating function*

$$P_G\left(\sum_{j=1}^{m} c_j, \sum_{j=1}^{m} c_j^2, \cdots, \sum_{j=1}^{m} c_j^r\right).$$

For a moment, let us return briefly to the problem of counting the total number of 2-colorings left fixed. This number is simply the sum of the coefficients in the pattern inventory. To sum coefficients, we simply set the indeterminants, b and w (and hence their powers), equal to 1, or equivalently, set $x_i = 2$ in P_G. If m colors were allowed, we would set $x_i = m$ in P_G, obtaining the same formula as in Corollary 1.

Finding the pattern inventories for the corner or face colorings of other unoriented regular m-gons is a good exercise, but practice has shown that only the most trivial 3-dimensional figures, such as a tetrahedron, are appropriate exercises. The cube can be treated in class with a tinker-toy model (without such a model, it is almost impossible to visualize the actions of all the motions). In doing such problems, there are two steps required before the formula can be used: first, a list of all the motions must be made (after some specific examples, the students should be able to deduce the size and contents of this list for an arbitrary regular m-gon); and second, the cycle structure representation of each motion must be determined (for a regular m-gon, this is equivalent to determining all the subgroups of the cyclic group of order m).

While the presentation here may seem fairly straightforward, it reads very differently from the standard treatment of Polya's formula. The importance of our approach is that theory and precise mathematical statements have been avoided in Sections 4 and 5 in favor of the underlying ideas that motivated Polya. For example, formally a coloring is a function $f: V \to C$ that assigns to each element $v \in V$ a color $c \in C$ and a permutation π leaves the coloring f fixed if $f(\pi^{-1}(v)) = f(v)$ for every v. For most students, such formalisms (and many are possible with this topic) make a derivation of this formula impossible to follow.

ON THE MINIMUM TRACK OF A MOVED LINE SEGMENT

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1. Introduction. The following unsolved problem in the calculus of variations is posed by Ulam [**1**]:

> Suppose two segments are given in the plane, each of length one. One is asked to move the first segment continuously without changing its length, to make it coincide at the end of the motion with the second given interval in such a way that the sum of the lengths of the two paths described by the end points should be a minimum. What is the general rule for this minimum motion?

This problem breaks down into two distinct problems. In the first problem, the sum of the motions of the ends of the line is to be minimized. In the second problem, the sum of the paths (roads or tracks) of the ends of the line is minimized, with the liberty of retracing parts of the track by either or both of the ends. In the second problem, the motion (or mileage) of the ends may be greater than the lengths of the tracks.

A solution of the restricted case in which no retracing is permitted, (that is, the motion equals the length of the track) was published by the author [**2**]. The present paper considers the second problem, namely, to find the least amount of track needed for the various positions of the final line with respect to the initial line.

These problems are reminiscent of the Kakeya minimal problem in which the area swept out by the moving unit line is to be minimized. Besicovitch [**3**] showed that there is no minimum for the Kakeya problem because the area can be as small as we wish. However, as the area is reduced to zero as a limit, the motion and tracks of the ends of the line both increase without limit. Cunningham [**4**] showed that the Besicovitch solution can be improved so that all the motion can be confined to a convex area which approaches a circle of unit radius as a limit.

2. The various cases. The nature of.the solutions of the minumum track problem depends upon several parameters of the terminal positions of the line, namely, their relative orientation, their separation, whether the terminal positions intersect, and whether the joins of the terminal positions of the line intersect.

Each of the various cases that is considered here is meant to illustrate a type of solution. Some of these cases admit different types of solutions, and each of these solutions must be evaluated to determine the one which yields the minimum track. It is always possible to transfer the line from its initial position to its final position by a pure rotation; but this method yields the minimum track only for the special cases in which the terminal lines are close.

Let the initial line AB, and the final line $A'B'$, satisfy the relation $AA' \geqq BB'$. In Cases V, VI and VII, let A'' be the third vertex of the equilateral triangle $AA'A''$

which includes AB and $A'B'$; and let B'' be the third vertex of the equilateral triangle $BB'B''$ which lies between AB and $A'B'$. Then, we have the following cases, as illustrated in the figures.

Case I. $ABB'A'$ is a short convex quadrilateral with alternate acute angles.

Case II. AB and $A'B'$ are widely separated.

Case III. AB and $A'B'$ cross, and $AA' < 1$.

Case IV. $ABB'A'$ is a convex quadrilateral with adjacent acute angles, and the angle between AB and $A'B'$ is less than $60°$.

Case V. $ABB'A'$ is a convex quadrilateral with adjacent acute angles, the angle between AB and $A'B'$ is greater than $60°$, and $A''B''$ does not cross AB or $A'B'$.

Case VI. $ABB'A'$ is a convex quadrilateral with adjacent acute angles, the angle between AB and $A'B'$ is greater than $60°$, and $A''B''$ crosses AB or $A'B'$.

Case VII. The angle between AB and $A'B'$ is greater than $60°$, $A''B''$ crosses both AB and $A'B'$, and $2 > AA' > 1$.

Case VIII. $ABA'B'$ is an intermediate convex quadrilateral.

Case IX. $ABA'B'$ is a short convex quadrilateral.

3. Notation and components of the track. In the figures, the track of the ends of the moving line is shown in dotted lines. The terminal positions of the line are shown in solid lines, except when they are part of the track. The dashed lines are construction lines or intermediate positions of the moving line.

If the total length of the track is designated by T, then it is given by

$$T = M + S + R,$$

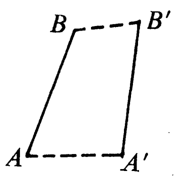
where M represents the main line, (AB', AA', BB' or $A'B$), connecting the end points of the terminal positions, S represents two units of siding for the given lines, and R represents portions of the track needed for rotation (the round house track).

For short movements, the only part that may be needed is M, and it may be composed of two pieces of track, one connecting A and A', and the other connecting B and B', as in Case I. For long movements, it may consist of only M and S, as shown in Case II, where nothing is added for rotation.

For rotations greater than $60°$, a shorter track is obtained by using a piece of siding to reduce the rotation to $60°$, as shown in Case V. The reduction in the length of the track, due to the reduced arc, is greater than the additional length of track needed for the siding. Note that the siding is traversed twice. Hence, the total length of track is reduced, although the motion may be greater than for some other path.

If the terminal positions are close, the shortest track is frequently composed entirely of rotation tracks of unit radius, as in Case III. Some are composed of siding chords and rotation arcs, as in Case IX.

4. Cases I and II. If A' is close to A, and B' is close to B, then the obvious shortest tracks are the straight lines joining these pairs, as shown in the figure for Case I. However, if the lines are widely separated, a shorter track is obtained by using part of the track as a common path for both points A and B. In Case IIa, the main line is BA'. The sidings EC and FC' are perpendicular to BA', pass through the points A and B' respectively, and are each of unit length. In making the transfer, the B-end goes from B to F, and then to C', while the A-end goes from A to C, then back through



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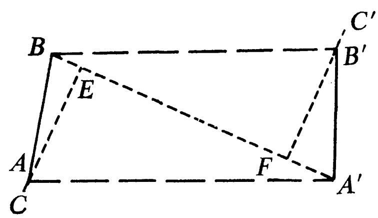
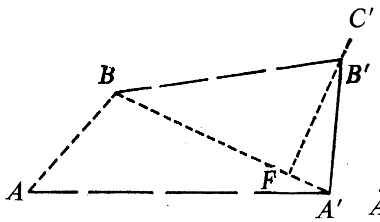
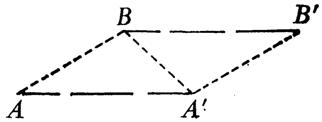


IIa



IIb



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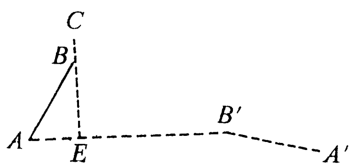

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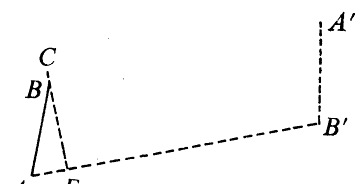
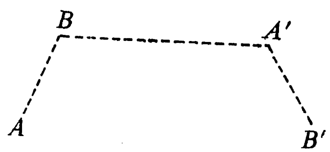


IIe



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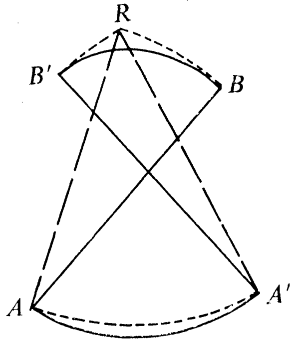
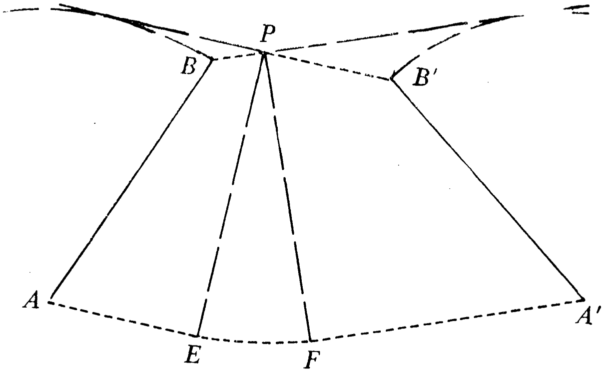
A to E, and then to F. As the A-end continues from F to A', the B-end retraces the segment $C'B'$ to return to B'. In Case IIb, the segment AC is not needed. In Case IIc, neither of AC and $B'C'$ is needed. Other typical cases are shown as Cases IId, IIe and IIf. Note that in all these cases, only straight sections of track were used.



III

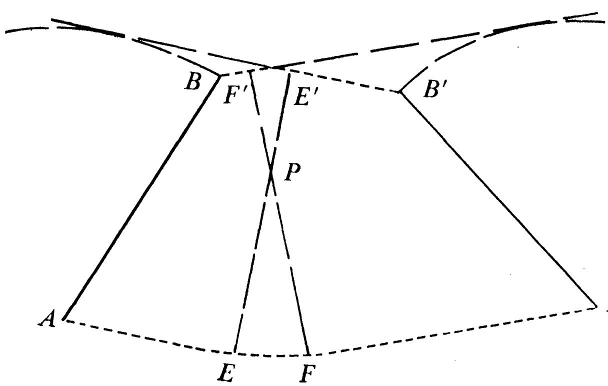
5. Case III. When AB and $A'B'$ cross each other, and AA' is less than unity, the shortest track is obtained by rotations. Rotation about a single point gives the track shown by the solid arcs of the figure. However, there are an infinite number of equivalent paths. One of these is shown by the dotted arcs of unit radius centered on A, A' and R. In general, any oval of constant unit width can serve, if we use arcs between any two diameters which make the same angle as AB and $A'B'$. Note that in this case, no straight sections of track are used.



IVa

6. Case IV. When the lines are close, and the angle between them is less than $60°$, and the acute angles are adjacent, then the paths of A and B have no part in common. In the figure for IVa, draw arcs with the acute vertices A and A' as centers. Draw tangents to these arcs from B and B'. Their intersection is the point P, which is the center of the arc EF'. Then the track of A is made of AE, EF and FA'. The track of B is BPB'. Then BP is parallel to FA' and PB' is parallel to AE.

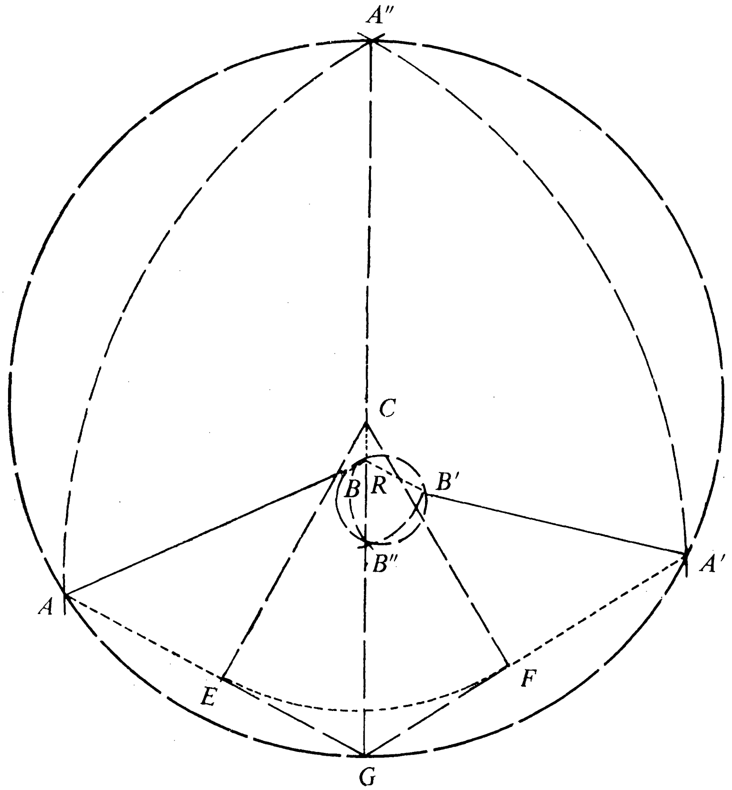
The derivation of the minimized track depends upon the application of what Riemann called the Dirichlet principle [5]. This is the use of a physical analogy which is assumed to simulate the mathematical problem. In the following derivations, a mechanical analogy and elementary statics are employed. If an elastic string, under uniform tension, is passed from P, through AB, to E and around the arc EF, to A' and through $A'B'$, and back to P, then the string assumes its shortest length at equilibrium. The turning sector PEF is considered as a rigid body. Then the sector is in equilibrium when the forces on it add to zero. This condition holds since the forces along PB and PB' have a resultant which is equal and opposite to the resultant of the forces along EA and FA'. This is similar to the minimum motion problem of the author's previous paper [2].



IVb

A variation to the track is obtained when the pivot point P of IVa is shifted to another point on the bisector of the angle BPB', as shown in the figure for IVb. The single arc EF of IVa is replaced by two arcs $E'F'$ and EF of equivalent total length, shown in IVb. Also, the sum of the straight segments of IVa is the same as the sum of the straight segments of IVb. Hence, the total track length of IVb is the same as the total track length of IVa. The curved portion of the track can be replaced by portions of the contour of any oval of constant unit width that is subtended between any two diameters making the same angle as EPF.

7. Case V. A more general case is shown as Case V. Here, the angle between the given lines is greater than 60°. Locate A'' as the third vertex of the equilateral triangle $AA'A''$ which contains the lines AB and $A'B'$. Locate B'' as the third vertex of the equilateral triangle $BB'B''$ which lies between AB and $A'B'$. Join A'' and B''. Then draw lines AE and $A'F$ which make angles of 60° with line $A''B''$. Locate C which is at unit distance from line AE and line $A'F$. Then C is the center of the arc

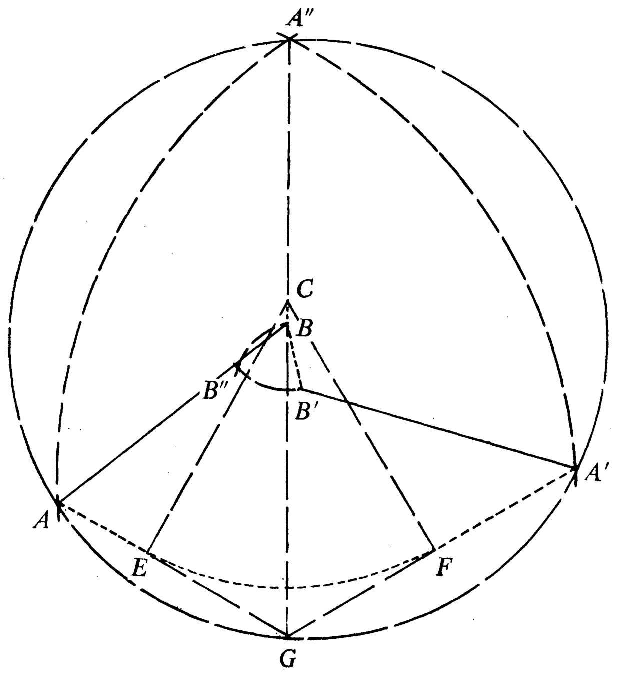


V

EF which is part of the track of point A. The other parts of the track of point A are AE and $A'F$. Draw BR and $B'R$ which make angles of 60° with line $A''B''$. The point R lies on the circle through $BB'B''$. Then the track of point B consists of the segments BR, CR and RB'. Note that R is the Fermat point of the triangle CBB'.

Again, the proof of this solution makes use of the mechanical analogy. Consider C as the center of a free rigid body which is a sector of unit radius. Then, an elastic string under uniform tension is fastened at C, passing through a free ring at R, through a fixed line AB, to E and around the arc EF, to A' and through the fixed line $A'B'$, and then back to the ring R where the other end of the string is fastened. Then, in equilibrium under uniform tension, the string assumes its shortest length. This length is the length of the tracks plus the two unit lengths of the given line. Since the tension in the segments RC, RB and RB' are equal, they must make equal angles with each other and the ring is in equilibrium. Also, since the tensions in AE and $A'F$ are at 60° with the line CR, the forces on the sector are in equilibrium.

8. Case VI. If the line $A''B''$ cuts AB (or $A'B'$), then use $A''B$ as the axis instead of $A''B''$. The line $A''B$ cuts the circle through $AA'A''$ in G. Draw lines GA and GA'. The arc of unit radius EF, which is tangent to these arcs, has its center at C. Then the elas-

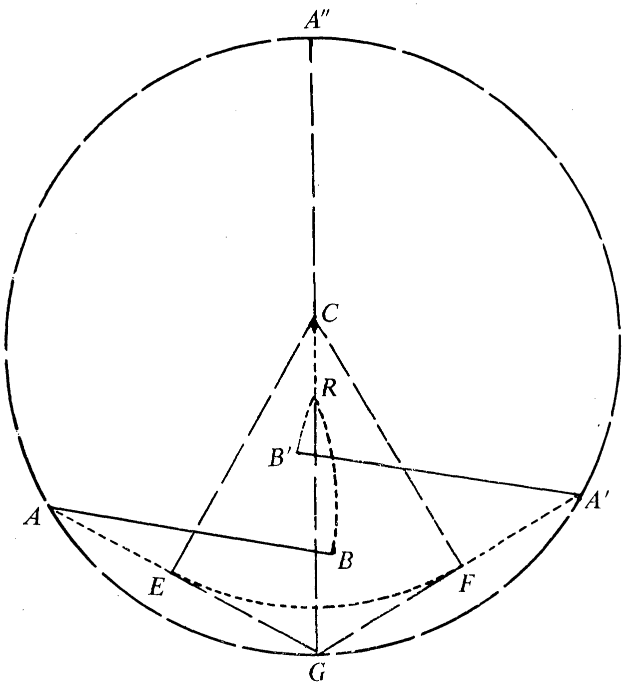


VI

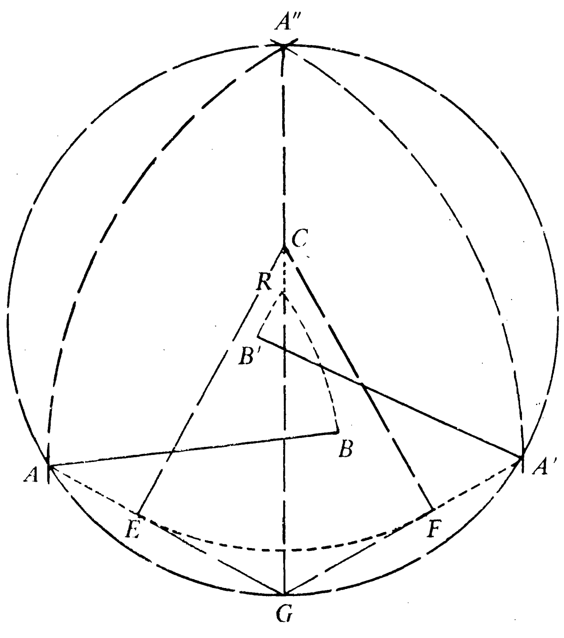
tic string fastened at C, through BA, to E and around the arc EF to A', through $A'B'$ and back to B where it is fastened, takes the shortest path when the sector CEF is in equilibrium. But, since the angles AGA'' and $A'GA''$ are each $60°$, the forces along EA and FA' are balanced by the force along CB acting on the sector at C, and the sector is in equilibrium.

9. Case VII. If AA' is greater than unity, then the oval of constant width cannot be used as in Case III. However, if $AA' < 2$, then arcs of unit radius centered on A and A' will intersect in R. The lines AB and $A'B'$ may be on the same side of AA' as in the figure for Case VIIa, or they may be on opposite sides as in the figure for Case VIIb. Find G, the intersection of $A''R$ with the circle through $AA'A''$. Draw lines AG and $A'G$. Draw the arc of unit radius tangent to these lines. If the center of this arc is designated by C, then the minimum track consists of the lines RC, RB, RB' and $AEFA'$. Using the string analogy, the force along CR is balanced by the forces along EA and FA'.

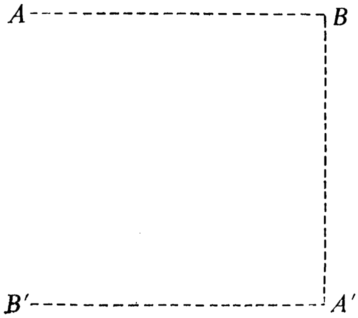
10. Case VIII. If $ABA'B'$ makes a convex quadrilateral that is not too long, then the shortest track is $ABA'B$, as shown in Case VIIIa. If AB and $A'B'$ are closer, then point P is found as in Case III to give the track $APBA'FEB'$. The closest case, shown as VIIIc, occurs when $AB' = BA' = 1/\sqrt{3}$ in the rectangle $ABA'B'$. The unsymmetric Case VIIId is somewhat more complicated. As the A-end goes to B', the B-end goes to



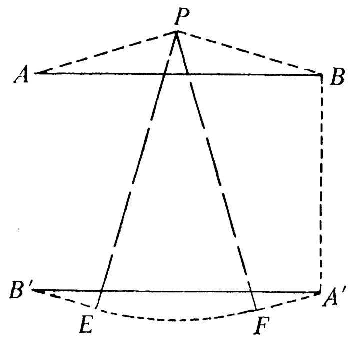
VIIa



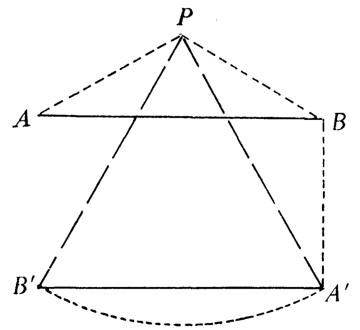
VIIb



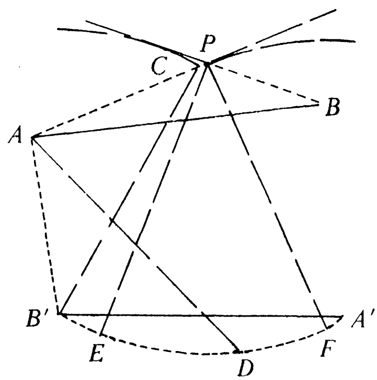
VIIIa



VIIIb



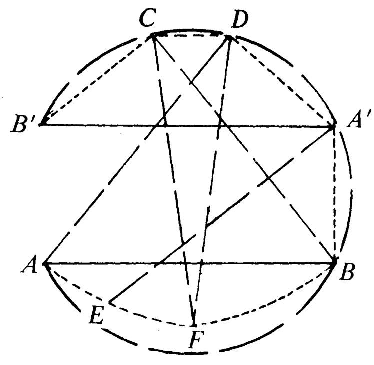
VIIIc



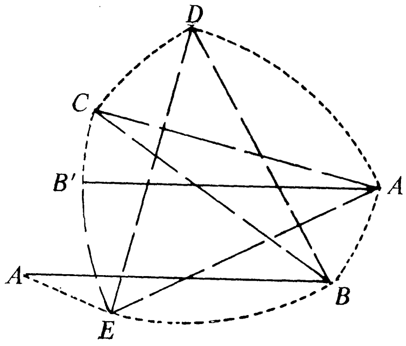
VIIId

P and C. As the A-end continues on the straight segment to E, the B-end returns to P. As the B-end is held stationary at P, the A-end traverses the arc EF. As the B-end continues from P to A, the A-end returns from F to D. Then, as the B-end completes its motion to B', the A-end retraces the arc DF and then the straight segment FA' to A'.

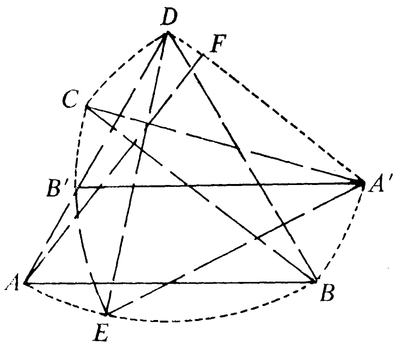
11. Case IX. If $ABA'B'$ makes a convex quadrilateral that is too small to use the construction of Case VIII, then the track may be composed entirely, or almost entirely, of chords and arcs of circles of rotation, as shown in Case IXa. As the A-end goes from $A'$to B, the B-end goes to C. While the B-end is stationary at C, the A-end goes from B to F. While the A-end is stationary at F, the B-end goes from C to D. While the B-end is stationary at D, the A-end goes from F to A. As the B-end goes from D to A', the A-end goes back from A to E. As the B-end goes from A' to B, the B-end returns from E to A.



IXa



IXb



IXc

Another method that may be used, if the lines AB and $A'B'$ are close, is shown in IXb. As the A-end is held stationary at A', the B-end moves from B' to C. Similarly, as the B-end continues to D and A, the A-end goes from A' to B and to E. Then, as the A-end moves along the straight segment EA, the B-end moves along the arc $A'B$ to terminate at B. Note that $EB'CDA'BE$ is a curve of constant width composed of arcs of unit radius. However, the track omits the arc EB', but adds the shorter straight segment EA.

Still another method, shown in IXc, may give still shorter tracks. The arc DA' of IXb is replaced by the shorter track segment DFA', while the straight segment EA of IXb is replaced by the arc EA of IXc.

12. Discussion of the results. There are transition cases bridging several of the cases considered in which the tracks undergo continuous change from one case to another, as in Cases V and VI. In some others, as in the transition between Cases I and II, there is a complete change in the character of the solutions. In some, as in Case III, there are an infinite number of different solutions with the same length.

In the tracks which are made of arcs of circles and straight line segments, no arc of unit radius should be greater than 60°. Instead, the longer arcs should be replaced by the combination of 60° arcs and straight line segments as illustrated in Cases V, VI and VII.

In some cases, the validity of the minimum is obvious. In others, the necessity of the balance of the forces in the string analogy gives an extremal, but this condition may not be sufficient. There may be other solutions which have not yet been found. Rigorous proofs are not available. A general rule for the solutions, demanded by the initial problem, does not seem to exist.

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5. Richard Courant and Herbert Robbins, What is Mathematics?, Oxford University Press, London, 1941, p. 368.

6. Problem E 2412, Amer. Math. Monthly, 81 (1974) 410.

UPPER BOUNDS ON THE MODULI OF THE ZEROS OF A POLYNOMIAL

R. C. RIDDELL, University of British Columbia

1. Introduction. According to a theorem of Cauchy [3, pp. 94–95], all the zeros of the complex polynomial

$$P(z) = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n$$

satisfy $|z| \leqq r_0$, where r_0 is the unique nonnegative solution of the equation

$$r^n - a_1 r^{n-1} - \cdots - a_{n-1} r - a_n = 0 \qquad (a_i = |\alpha_i|).$$

Since the bound r_0 is attained if $\alpha_i = -a_i$, it is the best possible one which is expressible in terms of the moduli of the coefficients, and all such bounds can be regarded as upper estimates on r_0. However, many explicit bounds have been obtained by applying eigenvalue location theorems to a companion matrix of $P(z)$, or by other special devices; and the relations of these bounds to one another, and to r_0, have been unclear.

In this brief exposition, we show first that several known bounds can be obtained directly and systematically from a suitable formulation of Cauchy's result. In the same spirit, we then establish some new bounds, which are of particular interest in case some modulus a_k is large in comparison to the other a_i. We conclude with a few numerical examples. The techniques are elementary, resting on little more than the

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I am indebted to J. L. Brenner for many useful discussions of this subject since the time I first met it in a seminar of his.

2. Some known bounds. Maintaining the notation $a_i = |\alpha_i|$ for the moduli of the coefficients of $P(z)$, and excluding the trivial case in which all $a_i = 0$, we begin by obtaining Cauchy's result in the following form.

THEOREM 2.1. *Suppose* $P(z) = 0$ *and* $r > 0$. *Then* $|z| \leqq M(r)$, *where*

$$M(r) = \max\{r, a_1 + a_2r^{-1} + \cdots + a_nr^{1-n}\}; \tag{1}$$

and hence $|z| \leqq M$ *if* $r = M$ *satisfies*

$$r \geqq a_1 + a_2r^{-1} + \cdots + a_nr^{1-n}. \tag{2}$$

REMARK. The right side of (2) is larger than r when $r > 0$ is small, and does not increase when r increases. Hence there is a unique $r_0 > 0$ which satisfies (2) with equality, and inequality (2) holds if and only if $r \geqq r_0$. Clearly r_0 is Cauchy's bound.

Proof. If $|z| \geqq r$, transpose all terms except z^n in the equation $P(z) = 0$, divide by z^{n-1}, and use the triangle inequality to obtain

$$\begin{aligned} |z| &\leqq a_1 + a_2|z|^{-1} + \cdots + a_n|z|^{1-n} \\ &\leqq a_1 + a_2r^{-1} + \cdots + a_nr^{1-n}. \end{aligned}$$

Thus either $|z| < r$, or else the last inequality above holds. This establishes the first assertion, from which the second follows immediately.

The bounds $M(r)$ of (1) have been established by Tikoo [**10** p. 689] by applying Gershgorin's eigenvalue location theorem to a general companion matrix of $P(z)$.

COROLLARY. *The zeros of* $P(z)$ *satisfy* $|z| \leqq M$, *where* M *is given by any of*

$$M = \max\{1, a_1 + a_2 + \cdots + a_n\}, \tag{3}$$

$$M = a_1 + a_2^{\frac{1}{2}} + \cdots + a_n^{1/n}, \tag{4}$$

$$M = a_1 + \frac{a_2}{a_1} + \cdots + \frac{a_n}{a_{n-1}}, \qquad (a_1 \cdots a_{n-1} \neq 0). \tag{5}$$

Proof. Bound (3) is (1) with $r = 1$. With r equal to the M of (4), $r \geqq a_i^{1/i}$ for $i = 1, \cdots, n$. Hence $r^{1-i} \leqq a_i^{1/i} \cdot a_i^{-1}$; and so the right side of (2) is no larger than the sum of the $a_i^{1/i}$, and (2) is satisfied. With r equal to the M of (5), $r \geqq a_j/a_{j-1}$ for $j = 1, \cdots, n$, where a_0 is taken to be 1. Multiplying these inequalities for $j = 1, \cdots, i-1$, we obtain $r^{i-1} \geqq a_{i-1}$, and so $a_ir^{1-i} \leqq a_i/a_{i-1}$, for $i = 1, \cdots, n$. Adding the last inequalities, we obtain (2).

Bound (3) is "classical" according to Parodi [**9**, p. 126, (V, 3)], and (4) is due to Walsh [**11**, p. 286]; but (5) does not seem to be in the literature.

THEOREM 2.2. *The zeros of $P(z)$ satisfy $|z| \leqq M(s)$ for any $s > 0$, where*

$$M(s) = \max\{s + a_1, s + a_2 s^{-1}, \cdots, s + a_{n-1}s^{2-n}, a_n s^{1-n}\}. \tag{6}$$

Proof. Suppose $s > 0$ is fixed, and set r equal to the number $M(s)$ in (6). To see that r satisfies (2), set

$$B = \max\{a_1, a_2 s^{-1}, \cdots, a_{n-1}s^{2-n}\},$$

so that $r \geqq B + s$ and $a_i \leqq Bs^{i-1}$ for $i = 1, \cdots, n-1$. Hence

$$\begin{aligned} &a_1 + a_2 r^{-1} + \cdots + a_{n-1}r^{2-n} \\ &\leqq B + \frac{Bs}{B+s} + \cdots + \frac{Bs^{n-2}}{(B+s)^{n-2}} \\ &= B + s - s\left(\frac{s}{B+s}\right)^{n-2}. \end{aligned} \tag{7}$$

Either $a_n s^{1-n} \leqq B + s$, in which case

$$r = B + s, \quad a_n \leqq s^{n-1}(B+s);$$

or else $a_n s^{1-n} > B + s$, in which case

$$r > B + s, \qquad a_n = s^{n-1}r;$$

and so in either case we find

$$a_n r^{1-n} \leqq s\left(\frac{s}{B+s}\right)^{n-2}.$$

By adding the last inequality to (7) and recalling that $B + s \leqq r$, we obtain (2).

The bounds $M(s)$ of (6), first established by Ballieu [**1**, p. 749], have been deduced from the Perron-Frobenius theorem by Wilf [**12**, p. 249] and from Gershgorin's theorem by Bell [**2**, p. 292]. The following particular cases should be compared with (3), (4), and (5).

COROLLARY. *The zeros of $P(z)$ satisfy $|z| \leqq M$, where M is given by any of*

$$M = \max\{1 + a_1, \cdots, 1 + a_{n-1}, a_n\}, \tag{8}$$

$$M = 2\max\{a_1, a_2^{\frac{1}{2}}, \cdots, a_n^{1/n}\}, \tag{9}$$

$$M = \max\left\{2a_1, \frac{a_2}{a_1} + a_1, \cdots, \frac{a_{n-1}}{a_{n-2}} + a_1, \frac{a_n}{a_{n-1}}\right\}, \quad (a_1 \cdots a_{n-1} \neq 0). \tag{10}$$

Proof. Bound (8) is (6) with $s = 1$. With s equal to the maximum of $a_i^{1/i}$ for $i = 1, \cdots, n$, we find $a_i s^{1-i} \leqq a_i^{1/i}$ just as in the proof of (4). Hence each item in (6) is dominated by $2s$, and (9) follows. To obtain (10), set

$$s = \max\left\{a_1, \frac{a_2}{a_1}, \cdots, \frac{a_{n-1}}{a_{n-2}}\right\}.$$

Then $s^{i-1} \geqq a_i/a_1$ for $i = 2,\cdots,n-1$, and

$$s^{n-1} = s \cdot s^{n-2} \geqq a_1(a_{n-1}/a_1) = a_{n-1};$$

hence $a_i s^{1-i} \leqq a_1$ for $i = 2,\cdots,n-1$, and $a_n s^{1-n} \leqq a_n/a_{n-1}$. Thus the right side of (6) is dominated by

$$\max\left\{s + a_1, \frac{a_n}{a_{n-1}}\right\},$$

from which (10) follows by the definition of s.

Bound (8) slightly sharpens Cauchy's first explicit bound [**3**, p. 92], (9) is due to Fujiwara [**5**, p. 168, (3)], and (10) slightly sharpens a bound due to Kojima [**6**, p. 121, (8)].

3. Some new bounds. From now on, k will denote an arbitrary but fixed integer in the range $1 \leqq k \leqq n$. We shall use the notations

$$a = a_k,\ b = a_1 + \cdots + a_{k-1},\ c = a_{k+1} + \cdots + a_n,$$

$$A = a_1 + \cdots + a_k + \cdots + a_n = a + b + c,$$

where it is understood that $b = 0$ if $k = 1$, and $c = 0$ if $k = n$.

THEOREM 3.1. *Assume $A \geqq 1$. Then the zeros of $P(z)$ satisfy $|z| \leqq M(t)$ for any $t > 0$, where*

$$M(t) = \max\left\{t + b, \left[a + \frac{1}{t}(ab + c)\right]^{1/k}\right\}. \tag{11}$$

Proof. The number $r = M(t)$ of (11) satisfies $r - b \geqq t$ and $r^k - a \geqq t^{-1}(ab + c)$. Multiplying these two inequalities together and dividing the result by r^k, we find

$$r \geqq b + ar^{1-k} + cr^{-k}. \tag{12}$$

Since $1 \leqq A = b + a + c$ by assumption, no number less than 1 satisfies (12). Thus $r \geqq 1$, and the right side of (12) dominates the right side of (2) in Theorem 2.1.

COROLLARY. *Assume $A \geqq 1$. Then the zeros of $P(z)$ satisfy $|z| \leqq M$, where M is given by either of*

$$M = \max\{1 + b, [a + ab + c]^{1/k}\}, \tag{13}$$

$$M = \tfrac{1}{2}(a^{1/k} + b) + \left[\tfrac{1}{4}(a^{1/k} - b)^2 + \frac{a^{1/k}}{ka}(ab + c)\right]^{\frac{1}{2}}, \qquad (a > 0). \tag{14}$$

Proof. Bound (13) is (11) with $t = 1$. To obtain (14) from (11), set

$$t = \tfrac{1}{2}x + [\tfrac{1}{4}x^2 + y]^{\frac{1}{2}},\ x = a^{1/k} - b,\ y = \frac{a^{1/k}}{ka}(ab + c),$$

and note that $t > 0$ since $a > 0$ is assumed. Since the graph of the kth root function lies on or below its tangent line at $(a, a^{1/k})$, the second member of (11) satisfies

$$\left[a + \frac{1}{t}(ab + c)\right]^{1/k} \leqq a^{1/k} + \frac{1}{t}y = x + b + \frac{1}{t}y.$$

But t satisfies the quadratic $xt + y = t^2$, and so

$$x + b + \frac{1}{t}y = t + b.$$

Thus $M(t)$ in (11) is equal to $t + b$, which, by the definition of t, is the M of (14).

In the case $k = 1$, both bounds in this corollary are known: (13) is just (3), and (14) has been deduced by Parodi [9, p. 126, (V, 5)] from an eigenvalue location theorem.

THEOREM 3.2. *Assume $A \leqq 1$. Then the zeros of $P(z)$ satisfy $|z| \leqq M(u)$ for any $u \geqq A^{1/k}$, where*

$$M(u) = \max\{u, [a + bu + cu^{k-n}]^{1/k}\}. \tag{15}$$

Proof. By assumption, $r = 1$ satisfies inequality (2) of Theorem 2.1. Hence there is nothing to prove if $M(u) > 1$, and we can assume $u \leqq M(u) \leqq 1$.

For $r > 0$, the function defined by

$$f(r) = a + br + cr^{k-n}$$

is convex, and so over the interval $u \leqq r \leqq 1$ it satisfies

$$f(r) \leqq \max\{f(u), f(1)\} \leqq M(u)^k.$$

In particular, $r = M(u)$ satisfies $r^k \geqq f(r)$, and hence

$$r \geqq br^{2-k} + ar^{1-k} + cr^{1-n}.$$

But since $r = M(u) \leqq 1$, the right side of the last inequality dominates the right side of (2).

COROLLARY. *Assume $A \leqq 1$. Then the zeros of $P(z)$ satisfy $|z| \leqq M$, where M is given by either of*

$$M = \max\{A^{1/k}, [a + bA^{1/k} + cA^{(n-k)/k}]^{1/k}\}, \tag{16}$$

$$M = \{\tfrac{1}{2}(a + b) + [\tfrac{1}{4}(a + b)^2 + c]^{\frac{1}{2}}\}^{1/p}, \qquad p = \max\{k, n - k\}. \tag{17}$$

Proof. Bound (16) is (15) with $u = A^{1/k}$. To obtain (17) from (15), set u equal to the M of (17), and note, by the hypothesis $A \leqq 1$, that the square root is at least $\frac{1}{2}(a + b) + c$, making $u \geqq A^{1/p} \geqq A^{1/k}$. The equation

$$u^p = a + b + u^{-p}, \qquad p = \max\{k, n - k\} \tag{18}$$

satisfied by u has no solution greater than 1, since $1 \geqq a + b + c$; and so $u \leqq 1$. This inequality together with (18) gives, in the two cases $k < n - k$ and $k \geqq n - k$, respectively:

$$u^k \geqq u^{n-k} = a + b + cu^{k-n} \geqq a + bu + cu^{k-n};$$

$$u^k = a + b + cu^{-k} \geqq a + bu + cu^{k-n}.$$

Hence in either case the $M(u)$ of (15) is equal to u.

In the case $k = n$, both (16) and (17) become $M = A^{1/n}$, valid when $A \leqq 1$. Combined with (3), this gives the bound $M = \max\{A, A^{1/n}\}$, to be found, e.g., in [8].

4. Numerical examples. Each of the following polynomials is accompanied by a list of upper bounds on the moduli of its zeros. Cauchy's bound r_0 is given first, followed by a table of upper estimates $M \geqq r_0$ computed by the indicated formulas in §2 and §3.

EXAMPLE 4.1.

$P(z) = z^6 + z^5 + 2z^4 + 3z^3 + 4z^2 + 5z + 6$; $r_0 = 2.61$

(3)	21.0	(13)	$k = 1$	21.0	(14)	$k = 1$	5.0
(4)	8.1		$k = 2$	4.7		$k = 2$	3.9
(5)	7.5		$k = 3$	4.0		$k = 3$	4.4
(8)	6.0		$k = 4$	7.0		$k = 4$	6.6
(9)	2.9		$k = 5$	11.0		$k = 5$	10.4
(10)	3.0		$k = 6$	16.0		$k = 6$	15.3

EXAMPLE 4.2.

$P(z) = z^6 + 2z^5 + 27z^3 + 1$; $r_0 = 3.84$

(3)	30.0	(13)	$k = 1$	30.0	(14)	$k = 1$	6.4
(4)	6.0		$k = 2$	5.3		$k = 2$	—
(5)	—		$k = 3$	4.4		$k = 3$	4.1
(8)	28.0		$k = 4$	30.0		$k = 4$	—
(9)	6.0		$k = 5$	30.0		$k = 5$	—
(10)	—		$k = 6$	30.0		$k = 6$	29.2

EXAMPLE 4.3.

$P(z) = z^6 + .20\, z^5 + .16\, z^4 + .12\, z^3 + .09\, z^2 + .06\, z + .03$; $r_0 = 0.87$

(3)	1.00	(16)	$k = 1$	3.88	(17)	$k = 1$	0.96
(4)	2.76		$k = 2$	1.01		$k = 2$	.94

(5) 3.67	$k=3$ 0.90	$k=3$ .90
(8) 1.20	$k=4$ .91	$k=4$ .92
(9) 1.14	$k=5$ .91	$k=5$ .93
(10) 1.00	$k=6$ .94	$k=6$ .94

EXAMPLE 4.4.

$P(z) = z^6 + .001\, z^4 + .216\, z^3 + .008;$ $r_0 = 0.63$

(3) 1.00	(16) $k=1$ 391.	(17) $k=1$ 0.93
(4) 0.98	$k=2$ 2.11	$k=2$ .85
(5) —	$k=3$ 0.64	$k=3$ .63
(8) 1.22	$k=4$ .69	$k=4$ .71
(9) 1.00	$k=5$ .75	$k=5$ .76
(10) —	$k=6$ .78	$k=6$ .78

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A NOTE ON THE GREATEST INTEGER FUNCTION

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1. Let $[x]$ denote the greatest integer less than or equal to the real number x. It is well known and easy to prove that (see for example [3, p. 97])

$$\sum_{r=0}^{k-1}\left[x+\frac{r}{k}\right]=[kx], \tag{1}$$

where k is an arbitrary positive integer. What can be said about the double sum

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left[x+\frac{r}{k}+\frac{s}{h}\right]? \tag{2}$$

To answer this and similar questions it is convenient to make use of the function [2, p. 1]

$$((x)) = \begin{cases} x-[x]-\frac{1}{2} & (x\neq \text{integer}) \\ 0 & (x=\text{integer}). \end{cases} \tag{3}$$

This function has the property

$$\sum_{r=0}^{k-1}\left(\left(x+\frac{r}{k}\right)\right) = ((kx)), \tag{4}$$

which in fact is equivalent to (1). Moreover $((x))$ is periodic with period 1:

$$((x)) = ((x+1)). \tag{5}$$

In view of (5), (4) can also be written in the form

$$\sum_{r(\text{mod } k)}\left(\left(x+\frac{r}{k}\right)\right) = ((kx)). \tag{6}$$

Now consider the sum

$$S=\sum_{r(\text{mod } k)}\sum_{s(\text{mod } h)}\left(\left(x+\frac{r}{k}+\frac{s}{h}\right)\right), \tag{7}$$

where h and k are relatively prime. When r runs through a complete residue system (mod k) and s a complete residue system (mod h), $hr+ks$ runs through a complete residue system (mod hk). Hence by (6),

$$S=\sum_{t(\text{mod } hk)}\left(\left(x+\frac{t}{hk}\right)\right)=((hkx))$$

so that

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left(\left(x+\frac{r}{k}+\frac{s}{h}\right)\right) = ((hkx)). \tag{8}$$

Assume, to begin with, that x is irrational. Then by (3) and (8), we have

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left\{x+\frac{r}{k}+\frac{s}{h}-\left[x+\frac{r}{k}+\frac{s}{h}\right]-\tfrac{1}{2}\right\} = hkx-[hkx]-\tfrac{1}{2},$$

so that

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left[x+\frac{r}{k}+\frac{s}{h}\right] = \sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left(x+\frac{r}{k}+\frac{s}{h}\right)-hkx+[hkx]-\tfrac{1}{2}hk+\tfrac{1}{2}. \tag{9}$$

Since

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left(x+\frac{r}{k}+\frac{s}{h}\right) = hkx+\tfrac{1}{2}h(k-1)+\tfrac{1}{2}k(h-1),$$

(9) reduces to

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left[x+\frac{r}{k}+\frac{s}{h}\right] = \tfrac{1}{2}(h-1)(k-1)+[hkx] \qquad ((h,k)=1). \tag{10}$$

We have assumed x irrational. But since the function $[x]$ is continuous from the right, it follows that (10) is valid for all real x. In particular, for $x=0$, we get

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left[\frac{r}{k}+\frac{s}{h}\right] = \tfrac{1}{2}(h-1)(k-1). \tag{11}$$

We remark that, by (1),

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left[x+\frac{r}{k}+\frac{s}{h}\right] = \sum_{r=0}^{k-1}\left[h\left(x+\frac{r}{k}\right)\right] = \sum_{s=0}^{h-1}\left[k\left(x+\frac{s}{h}\right)\right]. \tag{12}$$

The equality

$$\sum_{r=0}^{k-1}\left[h\left(x+\frac{r}{k}\right)\right] = \sum_{s=0}^{h-1}\left[k\left(x+\frac{s}{h}\right)\right] \tag{13}$$

is a known result [**1**].

The formula (10) can be extended slightly in the following way. Let

$$(a,k)=(b,h)=1 \tag{14}$$

as well as $(h,k)=1$. Then

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left(\left(x+\frac{ar}{k}+\frac{bs}{h}\right)\right) = \sum_{r=0}^{h-1}\sum_{s=0}^{k-1}\left(\left(x+\frac{r}{k}+\frac{s}{h}\right)\right) = ((hkx)).$$

Hence, as in the proof of (10), we get

$$\sum_{r=0}^{k-1}\sum_{s=0}^{h-1}\left[x+\frac{ar}{k}+\frac{bs}{h}\right] = \tfrac{1}{2}ah(k-1)+\tfrac{1}{2}bk(h-1)+\tfrac{1}{2}+[hkx]-\tfrac{1}{2}hk \tag{15}$$

for all real x.

2. In the next place, if we assume that

(16) $$(b,c) = (c,a) = (a,b) = 1,$$

then, exactly as above

(17) $$\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{t=0}^{c-1}\left(\left(x+\frac{r}{a}+\frac{s}{b}+\frac{t}{c}\right)\right) = ((abcx)).$$

Again, assume to begin with that x is irrational. Then, by (3), (17) gives

$$\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{t=0}^{c-1}\left[x+\frac{r}{a}+\frac{s}{b}+\frac{t}{c}\right]$$

$$= \sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{t=0}^{c-1}\left(x+\frac{r}{a}+\frac{s}{b}+\frac{t}{c}\right) - abcx + [abcx] - \tfrac{1}{2}abc + \tfrac{1}{2}$$

$$= \tfrac{1}{2}(a-1)bc + \tfrac{1}{2}(b-1)ca + \tfrac{1}{2}(c-1)ab + [abcx] - \tfrac{1}{2}abc + \tfrac{1}{2}.$$

Simplifying we get

(18) $$\sum_{r=0}^{a-1}\sum_{s=0}^{b-1}\sum_{t=0}^{c-1}\left[x+\frac{r}{a}+\frac{s}{b}+\frac{t}{c}\right] = abc - \tfrac{1}{2}(bc+ca+ab) + \tfrac{1}{2} + [abcx].$$

The restriction on x is removed as in (10).

Generally if $a_1, a_2, \cdots, a_n$ are relatively prime in pairs, we find that

(19) $$\sum_{r_i=0}^{a_i-1}\left[x+\frac{r_1}{a_1}+\cdots+\frac{r_n}{a_n}\right]$$

$$= \tfrac{1}{2}(n-1)a_1a_2\cdots a_n - \tfrac{1}{2}\sum_{i=1}^{n}\frac{a_1a_2\cdots a_n}{a_i} + \tfrac{1}{2} + [a_1a_2\cdots a_nx].$$

If in addition

(20) $$(c_i, a_i) = 1 \quad (i = 1,2,\cdots,n),$$

we have

(21) $$\sum_{r_i=0}^{a_i-1}\left[x+\frac{c_ir_1}{a_1}+\cdots\frac{c_nr_n}{a_n}\right]$$

$$= \tfrac{1}{2}\sum_{i=1}^{n} c_i(a_i-1)\frac{a_1a_2\cdots a_n}{a_i} - \tfrac{1}{2}a_1a_2\cdots a_n + \tfrac{1}{2} + [a_1a_2\cdots a_nx].$$

Finally we remark that (13) admits of considerable generalization. In particular we have such formulas as

(22) $$\sum_{s=0}^{b-1}\sum_{t=0}^{c-1}\left[a\left(x+\frac{s}{b}+\frac{t}{c}\right)\right] = \sum_{r=0}^{a-1}\sum_{t=0}^{c-1}\left[b\left(x+\frac{r}{a}+\frac{t}{c}\right)\right]$$

$$= \sum_{r=0}^{a-b}\sum_{s=0}^{b-1}\left[c\left(x+\frac{r}{a}+\frac{s}{b}\right)\right],$$

where of course

$$(b, c) = (c, a) = (a, b) = 1.$$

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ON MINKOWSKI'S THEOREM

P. R. SCOTT, University of Adelaide, South Australia

Let K be a closed convex set in the plane which is symmetric about the origin, and which has area $A(K)$. A well-known theorem of Minkowski asserts that if $A(K) \geqq 4$, then K contains a nonzero point of the integral lattice. We show that Minkowski's Theorem in the plane remains true for a large class of nonsymmetric sets.

THEOREM. *Let K be a closed convex set in the plane which contains the origin, O, as an interior point. Suppose that there exists a chord AOB of K which has midpoint O, and which partitions K into two disjoint regions having equal area. Then if $A(K) \geqq 4$, K contains a nonzero point of the integral lattice.*

Proof. Suppose that chord AOB partitions K into the two regions K_1, K_2. Let a, b be support lines to K at A, B respectively. If a, b are not parallel, we assume that they intersect on the K_1-side of AOB. Now let K_1' denote the reflection of K_1 in O, and set $K^* = K_1 \cup K_1'$. Then K^* is closed, convex and symmetric about O, and $A(K^*) \geqq 4$. It follows from Minkowski's theorem that K^* contains a nonzero point of the integral lattice. Using the symmetry of K^*, we deduce that K_1, and so K, contains a nonzero point of the integral lattice.

If we apply a nonsingular linear transformation T to the integral lattice, we obtain a general lattice Λ, with lattice determinant $d(\Lambda) = |\det T|$. Since such a transformation T preserves convexity, ratios of areas, and the property of being a midpoint, we immediately deduce the following more general result:

COROLLARY. *Let K satisfy the conditions of the theorem, and let Λ be a lattice in the plane. Then if $A(K) \geqq 4\, d(\Lambda)$, K contains a nonzero point of Λ.*

where of course

$$(b,c) = (c,a) = (a,b) = 1.$$

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A GEOMETRICAL INTERPRETATION OF CERTAIN ALGEBRAIC RESULTS

(In Memoriam: Thomas McSherry, 1948–1973)

DAN PEDOE, University of Minnesota

In my book (Pedoe [1]), I derive some geometrical consequences of the representation of the circles $X^2 + Y^2 - 2pX - 2qY + r = 0$ in the plane $Z = 0$ by the point (p, q, r) in E_3 spanned by the (X, Y, Z)-axes. The paraboloid of revolution: $\Omega: X^2 + Y^2 - Z = 0$ plays an important rôle in the discussion, its points being in one-to-one correspondence with the point-circles (circles of zero radius) in the plane $Z = 0$.

A coaxal system of circles in $Z = 0$ is represented by a line in E_3, and since a coaxal system is determined by the point-circles P, P' inverse in a circle Σ, the circle Σ belonging to the system, inversion in E_3 is represented thus: if V is the map of Σ, a line through V intersects Ω in points Q, Q' which are the respective maps of P, P' in $Z = 0$, regarded as point-circles.

The map in E_3 given by $Q \to Q'$ is an harmonic homology which extends to the whole of E_3, the polar plane of V with respect to Ω being the axial plane of the homology, and the point V being the center of the homology. Thomas McSherry asked whether this did not indicate a map in the large between circles in $Z = 0$. This map is not difficult to find, but curiously enough it is not specifically mentioned in the many books, including my own, which discuss inversion.

I discuss the circles which intersect three given circles $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$ at the same angle θ or $\pi - \theta$, and obtain the known result that these circles vary in four coaxal systems as we vary θ. These systems have one circle in common, the circle orthogonal to the three given circles. In E_3 the equations of the four lines which represent the four coaxal systems are simply:

$$\left.\begin{aligned} \sqrt{k_2}X_1 \pm \sqrt{k_1}X_2 = 0, \\ \sqrt{k_3}X_1 \pm \sqrt{k_1}X_3 = 0, \end{aligned}\right\}$$

where

$$k_i = p_i^2 + q_i^2 - r_i = (\text{radius of } \mathscr{C}_i)^2,$$

and $X_i = 0$ is the polar plane of (p_i, q_i, r_i) with respect to Ω, so that

$$X_i \equiv 2p_iX + 2q_iY - Z - r_i,$$

the point (p_i, q_i, r_i) representing the circle $\mathscr{C}_i$.

Thomas discovered a geometrical interpretation for these four lines in E_3. Let us denote by Π the plane through the three points (p_i, q_i, r_i), and let us call the intersection with Π of the plane through a line l in $Z = 0$ which is parallel to the Z-axis the *vertical projection* of l onto Π.

Then if the four *axes of similitude* ([1], p. 62, Ex. 13.2) in $Z = 0$ of the three circles $\mathscr{C}_i$ are projected vertically onto Π, and the polar lines of these projections taken with respect to Ω, these polar lines necessarily pass through the pole of Π (this point representing the unique circle orthogonal to the three given circles), and *it is these four lines which represent the four coaxal systems.*

From his result Thomas went on to deduce the Gergonne construction which gives a simple method for finding the eight circles which touch three given circles (see [**1**], Ex. 28.3, where the method is erroneously attributed to J. Casey). He also used his result to verify the theorem of a paper of mine [**2**] in which I show that three given circles cannot be specialized so as to possess only seven contact circles, the numbers 0, 1, 2, 3, 4, 5, 6 and 8 always being possible.

Thomas McSherry had given up a well-paid position in a computer firm to return to the University of Minnesota and train to become a high school teacher, when an apparent heart attack after a game of hockey killed him. Geometry is the poorer because of his untimely death.

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1. D. Pedoe, A Course of Geometry for Colleges and Universities, Cambridge University Press, 1970.
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WHEN DO THE PERIODIC ELEMENTS OF A GROUP FORM A SUBGROUP?

GARY J. SHERMAN, Rose-Hulman Institute of Technology

1. Introduction. An element x of the group G is said to be periodic if there exists a positive integer m for which $x^m = e$. If n is the smallest such positive integer, then x is said to be of order n. An element which is not periodic is said to be of infinite order. We denote the set of periodic elements in G by F and the set of elements of infinite order in G by I. It is assumed throughout this paper that $\{e\} \neq F$ and $I \neq \varnothing$. The subgroup generated by a subset H of G is denoted by $\langle H \rangle$.

One can geueralize the definition of periodicity as follows. Let S be a nonempty subset of G. An element x of G is said to be S-periodic if there are elements $g_1, \cdots, g_n$ in S for which

$$\prod_{i=1}^{n} g_i^{-1} x g_i = e.$$

If $S = \{e\}$, then S-periodicity is the usual notion of periodicity. If $S = G$, then S-periodicity is referred to as generalized periodicity. Generalized periodicity occurs naturally in the theory of partially ordered groups and provides the motivation for the definition of S-periodicity (for more on S-periodicity and connections with the theory of partially ordered groups see [**2**]).

While an obvious answer to the question raised in the title is: F is a subgroup if and only if F is closed under multiplication, another answer can be given in terms of taken with respect to Ω, these polar lines necessarily pass through the pole of Π (this point representing the unique circle orthogonal to the three given circles), and *it is these four lines which represent the four coaxal systems.*

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While an obvious answer to the question raised in the title is: F is a subgroup if and only if F is closed under multiplication, another answer can be given in terms of periodicity with respect to a set. Indeed, the main purpose of this note is to show that closure of F under multiplication is equivalent to the condition that every F-periodic element is periodic.

2. An example. Since $F \cup I = G$ and a group cannot be written as a union of two proper subgroups, it follows that one of F or I generates G, i.e., $\langle F\rangle = G$ or $\langle I\rangle = G$. In particular, if G is abelian, then F forms a proper subgroup and $\langle I\rangle = G$. Two questions arise naturally (invariably in my undergraduate modern algebra course) in view of the preceding remarks.

(1) Does the set of periodic elements of a group always form a subgroup?

(2) Does the set of elements of infinite order of a group always generate the group?

It is well known that the answer to question (1) is no. Since the answer to question (2) may not be generally known the following example is provided to show that both are answered in the negative.

Let $Z_2 * Z_2$ denote the free product of the cyclic group of order two with itself. A presentation for $Z_2 * Z_2$ is:

$$\langle a, b\colon a^2 = e \text{ and } b^2 = e\rangle.$$

The product ab is easily seen to be of infinite order and therefore F is not a subgroup. The reader can verify that $I \cup \{e\}$ consists of exactly those elements of even length. We note that each element of even length is its own inverse. Consider the product xy where x is of even length m and y is of even length n. If x ends and y begins with different letters, no cancellation takes place and xy has even length $m + n$. If x ends and y begins with the same letter, cancellation occurs until at least one of x or y is exhausted. In either event the length of xy is $|n - m|$, an even integer. We conclude that $I \cup \{e\}$ is a subgroup (in fact the cyclic subgroup generated by ab) and

$$\langle I\rangle = I \cup \{e\} \neq Z_2 * Z_2.$$

3. *F*-periodicity. We now use the notion of F-periodicity to provide conditions on G under which questions (1) and (2) have affirmative answers.

PROPOSITION 1. *The set of periodic elements of a group forms a subgroup if and only if every F-periodic element of the group is periodic.*

PROPOSITION 2. *If a group possesses an element which is not F-periodic, then the group is generated by the set of elements of infinite order.*

Whether the converse of Proposition 2 is valid or not is unknown. The author conjectures that it is false but is unable to provide a counterexample.

COROLLARY 1. *If a group is not generated by the set of elements of infinite order, then every element of the group is F-periodic.*

This corollary is just the contrapositive of Proposition 2 but it is worth stating since it indicates that a group may have elements of infinite order and yet be "periodic" to some degree. From the remarks in the introduction it follows that $Z_2 * Z_2$ is such a group.

The following two lemmas are useful in the proofs of Proposition 1 and Proposition 2:

LEMMA 1. *Let x and y be elements of the group G. If n is a positive integer, then $(xy)^n = x^n y'$ where y' is a product of conjugates of y. The conjugation is done by powers of x.*

Proof. We proceed by induction on n. The statement of the lemma is trivial when $n = 1$. From the induction hypothesis it follows that $(xy)^{k+1} = (xy)^k(xy) = (x^k y')(xy) = x^{k+1}(x^{-1}y'x)y$. Since conjugation is an automorphism, it follows immediately that $x^{-1}y'x$ is a product of conjugates of y by x when y' is.

LEMMA 2. *Let x and y be elements of the group G. If x is periodic of order k and y is not F-periodic, then xy and yx are of infinite order.*

Proof. If xy is periodic of order m, then $e = x^{km} = [(xy)y^{-1}]^{km}$. From Lemma 1 it follows that $e = [(xy)y^{-1}]^{km} = (xy)^{km}(y^{-1})' = (y^{-1})'$ where $(y^{-1})'$ is a product of conjugates of y^{-1} by powers of xy. This is a contradiction since y^{-1} is not F-periodic. Since xy and yx are conjugate they have the same order.

Proof of Proposition 1. Assume that F forms a subgroup of the group G. Let x be an F-periodic element of G, i.e., there are $g_1, \cdots, g_n$ in F for which $\prod_{i=1}^n g_i^{-1}xg_i = e$. It follows that $F = \prod_{i=1}^n g_i^{-1}xg_i\ F = \prod_{i=1}^n (g_i^{-1}F)\ (xF)\ (g_iF) = (xF)^n = x^nF$. Since x^n is periodic we conclude x is periodic.

Conversely, let a and b be periodic elements of G with orders m and n, respectively Putting $x = ab$, we get $a^{-1}x = b$. By Lemma 1, $e = b^{mn} = (a^{-1})^{nm}x' = x'$ where x' is a product of conjugates of x by a^{-1}. Since x is F-periodic, x is periodic and thus ab is periodic.

Proof of Proposition 2. If $\langle I\rangle \neq G$, then there exists an element x in $G - \langle I\rangle$. Let y be an element of G which is not F-periodic. Since $\langle I\rangle$ is a subgroup the product xy must be periodic. Lemma 2 implies xy is of infinite order. It follows that $\langle I\rangle = G$.

References

1. J. J. Rotman, The Theory of Groups: An Introduction, Allyn & Bacon, Boston, 1965.
2. G. J. Sherman, Doctoral Thesis, Indiana University, 1971.

AN INEQUALITY FOR CONDITIONAL DISTRIBUTIONS

B. R. JOHNSON, University of Victoria

Consider the problem of comparing the variance of a random variable X with its conditional variance, given X is known to have taken a value in the interval I. At first glance it would appear that this additional information about X would serve to reduce the variance. However, this observation is false in general. For example, suppose X may assume one of the three possible values $-1, 0, 1$ with probabilities $p, p, 1-2p$

The following two lemmas are useful in the proofs of Proposition 1 and Proposition 2:

LEMMA 1. *Let x and y be elements of the group G. If n is a positive integer, then $(xy)^n = x^n y'$ where y' is a product of conjugates of y. The conjugation is done by powers of x.*

Proof. We proceed by induction on n. The statement of the lemma is trivial when $n = 1$. From the induction hypothesis it follows that $(xy)^{k+1} = (xy)^k(xy) = (x^k y')(xy) = x^{k+1}(x^{-1}y'x)y$. Since conjugation is an automorphism, it follows immediately that $x^{-1}y'x$ is a product of conjugates of y by x when y' is.

LEMMA 2. *Let x and y be elements of the group G. If x is periodic of order k and y is not F-periodic, then xy and yx are of infinite order.*

Proof. If xy is periodic of order m, then $e = x^{km} = [(xy)y^{-1}]^{km}$. From Lemma 1 it follows that $e = [(xy)y^{-1}]^{km} = (xy)^{km}(y^{-1})' = (y^{-1})'$ where $(y^{-1})'$ is a product of conjugates of y^{-1} by powers of xy. This is a contradiction since y^{-1} is not F-periodic. Since xy and yx are conjugate they have the same order.

Proof of Proposition 1. Assume that F forms a subgroup of the group G. Let x be an F-periodic element of G, i.e., there are $g_1, \cdots, g_n$ in F for which $\prod_{i=1}^n g_i^{-1}xg_i = e$. It follows that $F = \prod_{i=1}^n g_i^{-1}xg_i \; F = \prod_{i=1}^n (g_i^{-1}F) \; (xF) \; (g_iF) = (xF)^n = x^nF$. Since x^n is periodic we conclude x is periodic.

Conversely, let a and b be periodic elements of G with orders m and n, respectively Putting $x = ab$, we get $a^{-1}x = b$. By Lemma 1, $e = b^{mn} = (a^{-1})^{nm}x' = x'$ where x' is a product of conjugates of x by a^{-1}. Since x is F-periodic, x is periodic and thus ab is periodic.

Proof of Proposition 2. If $\langle I\rangle \neq G$, then there exists an element x in $G - \langle I\rangle$. Let y be an element of G which is not F-periodic. Since $\langle I\rangle$ is a subgroup the product xy must be periodic. Lemma 2 implies xy is of infinite order. It follows that $\langle I\rangle = G$.

References

1. J. J. Rotman, The Theory of Groups: An Introduction, Allyn & Bacon, Boston, 1965.
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Consider the problem of comparing the variance of a random variable X with its conditional variance, given X is known to have taken a value in the interval I. At first glance it would appear that this additional information about X would serve to reduce the variance. However, this observation is false in general. For example, suppose X may assume one of the three possible values $-1, 0, 1$ with probabilities $p, p, 1-2p$

respectively, where $0 < p < \frac{1}{2}$. Then $\mathrm{Var}(X) \to 0$ as $p \to 0$, while $\mathrm{Var}(X \mid X \in [-1,0])$ $= \frac{1}{4}$ regardless of p. Thus, there are many values of p for which

$$\mathrm{Var}(X) < \mathrm{Var}(X \mid X \in [-1,0]).$$

On the other hand, the anticipated reduction in variance does occur whenever the mean remains unchanged after conditioning.

LEMMA. *If the conditional probability distribution, given an interval, has the same mean as the original distribution, then the variance of the conditional distribution is less than or equal to the variance of the original distribution.*

This intuitively obvious lemma provides motivation to consider other general properties relating a conditional probability distribution to its original distribution. The following theorem is an outgrowth of such considerations:

THEOREM. *Let P be a probability measure on the Borel subsets of the real line $\mathscr{R}$, and let P_I denote the conditional distribution of P, given the interval I. If f is a nonnegative, monotonic, integrable function such that*

$$\int f dP_I = \int f dP,$$

then

$$\int f^r dP_I \leqq \int f^r dP \text{ for all } r > 1. \tag{1}$$

Furthermore, this inequality is strict iff f is strictly monotonic and $P(I) < 1$.

COROLLARY. *Suppose X is a nonnegative valued random variable with finite expectation, and I is an interval such that*

$$E(X \mid X \in I) = E(X).$$

Then

$$E(X^r \mid X \in I) < E(X^r) \text{ for all } r > 1,$$

unless $P(X \in I) = 1$.

Proof of theorem. Formal proof will be given only for the case where I is a finite interval such that $P(I) < 1$, f is positive valued, and $\int f^r dP < \infty$ for all r. Routine modifications are required for the remaining cases.

Define

$$\phi(r) = \int f^r dP_I \Big/ \int f^r dP.$$

We must show that $\phi(r) \leqq 1$ for all $r > 1$. Since $\phi(1) = 1$, it suffices to show that

$\phi(r)$ is nonincreasing on the interval $[1, \infty)$. Therefore, by the mean value theorem, it suffices to show that $\phi'(r) \leqq 0$ for all $r > 1$.

Using Lebesgue's dominated convergence theorem to justify differentiating under the integral, and recalling that

$$\frac{d}{du} a^u = a^u \log a,$$

we obtain

$$\phi'(r) = \frac{C(r)}{P(I)} \left\{ \iint_{AI} + \iint_{BI} \right\} H_r(x, y)\, dP(x)\, dP(y)$$

where

$$C(r) = \left(\int f^r\, dP \right)^{-2},$$

$$H_r(x, y) = [f(x)f(y)]^r [\log f(x) - \log f(y)],$$

$$A = \{y \colon y < x \text{ for all } x \in I\},$$

$$B = \{y \colon y > x \text{ for all } x \in I\}.$$

Since $\phi(0) = \phi(1) = 1$, we may apply the mean value theorem to find $r^* \in (0, 1)$ such that $\phi'(r^*) = 0$. Letting a and b denote respectively the left-hand and right-hand end points of the interval I, and using the monotonicity of f, we have for $t > 0$

$$[f(x)f(y)]^t \leqq (\geqq)\ [f(b)f(a)]^t \text{ and } H_{r^*}(x, y) \geqq (\leqq) 0$$

for all $x \in I$, $y \in A$ provided f is nondecreasing (nonincreasing). Therefore,

$$[f(x)f(y)]^t H_{r^*}(x, y) \leqq [f(b)f(a)]^t H_{r^*}(x, y) \text{ for all } x \in I,\ y \in A.$$

Similarly, this inequality is found to be valid for all $x \in I$, $y \in B$ as well. Hence, we obtain for $t > 0$

$$\begin{aligned} \phi'(r^* + t) &= \frac{C(r^* + t)}{P(I)} \left\{ \iint_{AI} + \iint_{BI} \right\} [f(x)f(y)]^t H_{r^*}(x, y)\, dP(x)\, dP(y) \\ &\leqq \frac{C(r^* + t)}{P(I)} \left\{ \iint_{AI} + \iint_{BI} \right\} [f(a)f(b)]^t H_{r^*}(x, y)\, dP(x)\, dP(y) \\ &= \frac{C(r^* + t)}{C(r^*)} [f(a)f(b)]^t \phi'(r^*) = 0. \end{aligned}$$

Thus, inequality (1) is established. The strictness of this inequality for strictly monotonic f follows from the fact that

$$\phi'(r^* + t) < 0 \text{ for all } t > 0$$

in this situation.

BOOK REVIEWS

EDITED BY ADA PELUSO AND WILLIAM WOOTON

Materials intended for review should be sent to: Professor Ada Peluso, Department of Mathematics, Hunter College of CUNY, 695 Park Avenue, New York, New York 10021, or to Professor William Wooton, 1495 La Linda Drive, Lake San Marcos, California 92069. A boldface capital C in the margin indicates that a review is based in part on classroom use.

Statistics: A Guide to the Unknown. Edited by Judith M. Tanur, Frederick Mosteller, et al. Holden-Day, San Francisco, California, 1972. xxiii + 430 pp. $5.95.

With the lessened availability of funding for mathematical research and the related decline in the employability of Ph.D.-holding mathematicians have come numerous expressions of concern over the public image of the professional mathematician. In a recent *Newsletter* of the Conference Board of the Mathematical Sciences, Truman Botts, Executive Director of the Conference Board, urges mathematicians to accept the challenge to write effectively about their subject at the popular level. He cites a few instances where mathematicians have successfully written for the layman, but he emphasizes the difficulty in doing so and the rarity with which it has been done.

Statistics: A Guide to the Unknown represents a major effort designed to meet this challenge in the areas of statistics and probability. The *Guide* is the result of a project initiated by the Joint Committee on the Curriculum in Statistics and Probability of the American Statistical Association and the National Council of Teachers of Mathematics. According to its editors, the book is aimed at segments of the lay public: parents of school children, school superintendents, principals, board members, teachers of mathematics, and finally, students of mathematics. As the editors state, this paperback is not a textbook, but it could serve as an auxiliary text for statistics courses at different levels.

The *Guide* is a collection of essays, by experts in various specializations, which describe in nonmathematical language diverse applications of statistics and probability. The essays are short (averaging less than ten pages each), readable, interesting, and informative. The book's utility is enhanced by an extensive index, and by a table of contents which is arranged by subject, but which includes authors and short abstracts. Following the table of contents are classifications of the articles by data sources and by statistical tools. These features will be welcomed by those who may use the book as a supplemental text in a statistics course. Of further merit is the cross-referencing that often occurs within the articles themselves. For example, in Louis J. Battan's article, "Cloud Seeding and Rainmaking," when the notion of correlation of the average precipitation of two geographical areas is introduced, the reader is provided a one-sentence summary of the meaning of correlation. He is then referred to a more detailed, yet intuitive, discussion of this concept in the article by C.A. Whitney which deals with correlations between solar brightness measurements.

The high quality of the essays is perhaps equalled by the variety of uses of statistics and probability which they expose. An essay by Hans Zeisel and Harry Kalven, Jr. explores recent applications of statistics and probability within the legal domain. A statistical analysis is outlined which demonstrates the high probability that the jury selection for the 1968 trial of Dr. Benjamin Spock involved systematic discrimination against women. In his essay, Nathan Keyfitz investigates models for population growth and uses them to project populations of various countries. I especially liked the approach of Keyfitz and some other authors in *doing* a bit of mathematics in essays which are largely descriptive.

While this reviewer thinks that the *Guide* is undeniably successful in informing its reader in a lively manner of the fascinating uses of statistics, my estimate of its potential success as an auxiliary textbook in a statistics course is somewhat qualified. As a supplement to a traditional text in an introductory course in probability and statistics, the *Guide* will perhaps inform, entertain, and even fascinate the student. However, by its very design it will teach him little mathematics unless the essays are supplemented by explanatory lectures designed to relate the material mathematically to the main body of the course. Such an effort could undoubtedly be fruitful, but would be costly in time expended, and thus would likely be neglected by the instructor. The book is a welcome source from which the student can glean much knowledge *about* statistics, but it is not one from which he will learn statistics. Nor could it have been and yet so magnificently have achieved its stated objectives.

J. D. EMERSON, Middlebury College

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before June 1, 1975.

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PROPOSALS

915. *Proposed by Jerome H. Manheim, Bradley University, Peoria, Illinois*

For arbitrary k and p show the bracket function satisfies

$$\left[\frac{10^k - p}{p}\right] + \left[\frac{-(10^k + 1)}{p}\right] + 2 = 0.$$

916. *Proposed by H. Demir, M.E.T.U., Ankara, Turkey.*

Let XYZ be the pedal triangle of a point P with regard to the triangle ABC. Then find the trilinear coordinates x, y, z of P such that

$$YA + AZ = ZB + BX = XC + CY.$$

917. *Proposed by Charles W. Trigg, San Diego, California.*

The length of every edge of a regular pentagonal prism is e.

(a) When the two pentagonal faces are rotated about parallel diagonals until two of their edges coincide, two lateral edges vanish and one becomes elongated. The resulting hexahedron has two congruent regular pentagons, two congruent equilateral triangles, and two congruent trapezoids for faces. Eleven of its edges are equal. What is the length of the twelfth edge?

(b) When the pentagonal faces are otherwise rotated about the parallel diagonals until two of the vertices coincide, one lateral edge vanishes and two are elongated. The resulting heptahedron has two congruent regular pentagons, two congruent equilateral triangles, two congruent trapezoids and one rectangle for faces. Twelve of its edges are equal. What are the lengths of the other two edges?

918. *Proposed by Erwin Just, Bronx Community College.*

Let F_n be the nth member of the sequence defined by $F_n = F_{n-2} + F_{n-3}$, with $F_1 = 1$, $F_2 = 0$, $F_3 = 1$. Prove that $F_{2n} - F_{n-1}^2$ is divisible by F_n.

919. *Proposed by M. S. Klamkin, Ford Motor Company.*

An $(n+1)$-dimensional simplex with vertices $O, A_1, A_2, \cdots A_{n+1}$ is such that the $(n+1)$ concurrent edges OA_i are mutually orthogonal. Show that the orthogonal projection of O onto the n-dimensional face opposite to it coincides with the orthocenter of that face (this generalizes the known result for $n = 2$).

920. *Proposed by Leon Bankoff, Los Angeles, California.*

If r, r_1, r_2, r_3 are the inradius and ex-radii of a triangle and h_1, h_2, h_3 are the altitudes, show that the radius of the nine-point circle is equal to $rr_1r_2r_3/h_1h_2h_3$.

921. *Proposed by Heiko Harborth, TU Braunschweig, Germany.*

Determine all integers n with $n = a^b/[\phi(a^b) - (\phi(a))^b]$, where a and b are positive integers, and $\phi(m)$ is Euler's quotient function.

Errata. In Problem 904, page 166, May-June 1974, the second line should read as follows:

"... is both minimum in its row and minimum in its column. Do this..."

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 603. Find all twice differentiable functions f such that for all x,

$$f'(x) = f(-x).$$

[*Submitted by Peter A. Lindstrom*]

Q 604. Prove that $\sin 10° \sin 30° \sin 50° \sin 70° = 1/16$.

[*Submitted by Zalman Usiskin*]

Q 605. Show that if x, y, z is a Pythagorean triplet, then $2xy < z^2$.

[*Submitted by Norman Schaumberger*]

Q 606. Show elementarily that

$$(x + y + z)^{x+y+z} \geqq x^x y^y z^z$$

for positive x, y, z.

[*Submitted by Murray Klamkin*]

Q 607. Prove that there exists an everywhere dense set of planar points in which the distance between each pair of points of the set is irrational.

[*Submitted by Erwin Just*]

(Answers on page 297.)

SOLUTIONS

Late Solutions

Also solved by Joe Albree, University of New Orleans, **879**; *Theresa Bevacqua, Southern Connecticut State College, New Haven, Connecticut,* **882**; *Steve Dillard, Westmont College, Santa Barbara, California,* **880**; *Ragnar Dybvik, Rektor, N-6630 Tingvoll, Norway,* **878**; *Padmini T. Joshi, Ball State University, Muncie, Indiana,* **882**; *C. B. A. Peck, State College, Pennsylvania,* **880**; *Karl Heuer, Turnich, West Germany,* **885**; *Steve Perry, Westmont College, Santa Barbara, California,* **882**; *Romesh Singh, Arthur District High School, Arthur, Ontario, Canada,* **880, 882**; *G. A. Heuer, University Zu Koln, Weyertal, West Germany,* **882, 883**; *Harry Whitcomb, Philadelphia College of Pharmacy and Science, Philadelphia, Pennsylvania,* **880**.

Opposite in Sign

887. [January 1974] *Proposed by Norman Schaumberger, Bronx Community College.*

One solution of the equation $x^y y^x = 1$ is $x = 4$, $y = -2$. Show that all unequal rational solutions are opposite in sign.

Solution by M. G. Greening, University of New South Wales, Australia.

(i) Assume $x > y > 0$. Set $x = a/b$, $y = c/d$ with $(a,b) = 1 = (c,d)$.

Then $a^{bc}c^{ad} = b^{bc}d^{ad}$ and necessarily (α) $a^{bc} = d^{ad}$, (β) $c^{ad} = b^{bc}$ as $a = b = 1$ (or $c = d = 1$) leads to $x = y = 1$.

If $p \mid a$ then, for some $r > 1$, $p^r \mid a$, $p^{r+1} \nmid a$ while $p^s \mid d$ and $p^{s+1} \nmid d$ for some $s > 1$, resulting in $rbc = sad$. It follows that $(ad, bc) = 1$ so that $r = ad$, $s = bc$ and (γ) $r \geqq p^{r+s}$; (δ) $s \geqq q^{r+s}$ where $q \mid b$ and necessarily $q \mid c$.

Now $r = s = 1$ is impossible as then $x = y = 1$, so, at most one of r, $s = 1$. But if m is an integer, $m^{t+1} > t$ unless $m = 1 = t$, and the simultaneous solution of (γ), (δ) is impossible under the stated conditions, whence (1) $x > y > 0$ and (2) $x^y y^x = 1$ is impossible for x, y rational.

(ii) Assume $x < y < 0$. Either x^y and y^x are both positive or both negative. If both positive then $x^y = |x|^y$ and $y^x = |y|^x$, while if both negative $x^y = -|x|^y$, $y^x = -|y|^x$ and $x^y y^x = |x|^y |y|^x$.

But $|x|^y |y|^x = (|x|^{|y|} |y|^{|x|})^{-1}$ so that $x^y y^x = 1$ demands $|x|^{|y|} |y|^{|x|} = 1$ with $|x| > |y| > 0$ as in (i).

Therefore x, y of same sign is impossible, if $x^y y^x = 1$, and the conclusion follows.

Also solved by Mary G. Checco; Thomas E. Elsner, General Motors Institute, Flint, Michigan; Robert M. Hashway, Bronx Community College; Aron Pinker, Frostburg State College, Frostburg, Maryland; Long Sang Chiu and Man Keung Siu (jointly), University of Miami, Coral Gables, Florida; Norman Schaumberger, Bronx Community College, New York; R. S. Stacy, Manzano High School, Albuquerque, New Mexico; Kenneth M. Wilke, First National Bank of Topeka Bldg., Kansas, and the proposer.

A Computation

888. [January 1974]. *Proposed by E. M. Clarke, Madison College, Harrisonburg, Virginia.*

Let G be defined by

$$G = \begin{cases} x - 10 & \text{if } x > 100 \\ G(G(x+11)) & \text{if } x \leqq 100. \end{cases}$$

Compute $\int_0^{100} G(x)dx$.

Solution by Richard A. Gibbs, Fort Lewis College, Colorado.

Let $G^k(x) = G(G^{k-1}(x))$, $k = 2, 3, \cdots$. Choose $x \in [0, 99]$ and let n be the smallest positive integer for which $x + 11n > 100$. Let $S = \{r : 11k < r \leqq 11k + 1, k = 0, 1, \cdots, 8\}$.

There are two cases:

(1) If $x \notin S$, then $100 < x + 11n \leqq 110$ and

$$G(x) = G^2(x + 11) = \cdots = G^{n+1}(x + 11n) = G^n((x + 1) + 11(n-1)) = \cdots = G(x + 1).$$

(2) If $x \in S$, then $110 < x + 11n \leqq 111$ and

$$G(x) = \cdots = G^{n+1}(x + 11n) = G^n((x + 1) + 11(n-1)) = G^{n-1}((x + 2) + 11(n-2)) = \cdots = G(x + 2).$$

However, since $x + 1 \notin S$, it follows from case (1) that $G(x + 1) = G(x + 2)$. Therefore $G(x) = G(x + 1)$ for all $x \in [0, 99]$ and G is periodic of period 1. It is easy to show that $G(99 + \delta) = 90 + \delta$ for $0 < \delta \leqq 1$. Thus

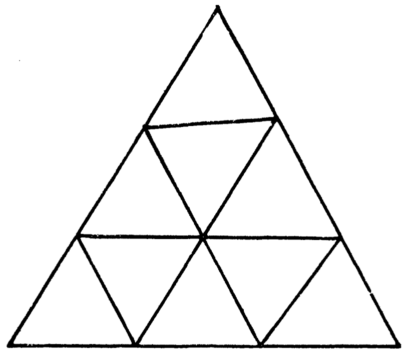
$$\int_0^{100} G(x)dx = 100 \int_{99}^{100} G(x)dx = 9050.$$

Also solved by William B. Adams, Boston University, Boston, Massachusetts; Carl A. Argila, De La Salle College, Manila, Philippines; Gladwin Bartel, Otero Junior College, La Junta, Colorado; Dr. J. C. Binz, Bolligen, Switzerland; Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania; Richard L. Breisch, Alamogordo, New Mexico; Joseph B. Browne, Oklahoma State University; F. A. Chimenti, State University College, Fredonia, New York; Brian Conery, University of Santa Clara, California; D. P. Choudhury, I. I. T., Kanpur, India; Romae J. Cormier, DeKalb, Illinois; Stephen C. Currier, Jr. Penn. State University, Altoona, Pennsylvania; Santo M. Diano, Havertown, Pennsylvania; Thomas E. Elsner; Abraham L. Epstein, Air Force Cambridge Research Laboratories, Bedford, Massachusetts; Michael Goldberg, Washington, D. C.; M. G. Greening, University of New South Wales, Kensington, Australia; Dr. Heiko Harborth, TU Braunschweig, Germany; Kent Harris, Western Illinois University, Macomb, Illinois; E. M. Clarke, Madison College, Harrisonburg, Virginia; Karl Heuer, Turnich, West Germany; Richard A. Jacobson, Houghton College, New York; Henry S. Lieberman, John Hancock Mutual Life Insurance Company Boston, Massachusetts; Kay P. Litchfield, Provo, Utah; Mary Helen Manning, Villanova University; Robert A. Meyer, University of Nebraska, Lincoln, Nebraska; Joseph V. Michalowig; Maurice M. Mizrahi, University of Texas, Austin, Texas; James W. McHutchion, Ohio State University, Columbus, Ohio; William Nuesslein, New York State Insurance Department, Albany, New York; Lawrence A. Ringenberg, Eastern Illinois University, Charleston, Illinois; Long Sang Chiu and Man Keung Siu (jointly), University of Miami, Coral Gables, Florida; University of Santa Clara Problem Solving Seminar, University of Santa Clara; Paul Shimp, University of New Orleans; Joseph Silverman, Brown University, Providence, R. I.; Roland Smith, Russell Sage College, Troy, New York; R. S. Stacy, Manzano High School, Albuquerque, New Mexico; Phil Tracy, Liverpool, New York; Harold Ziehms, Naval Postgraduate School, Monterey, California; Gene Zirkel, Nassau Community College, Garden City, New York; A. Zujus; E. M. Clarke, Harrisonburg, Virginia, and the proposer.

A Well-Known Problem

889. [January 1974] *Proposed by Ralph E. Edwards, Baltimore Life Insurance Company, Maryland.*

In the accompanying diagram, $f(x)$, the different number of triangles, is 13, where x, the number of horizontal lines, is 3. Find the formula for $f(x)$.



1. *Comment by Bob Prielipp and N. J. Kuenzi, Oshkosh, Wisconsin.*

A recent discussion of this problem is found in the article *The number of triangles in a triangular lattice* by J. W. Moon and N. J. Pullman. This article appears on pages 28–31 in the Supplement to Vol. 3, No. 4, Fall 1973 issue of *Delta*. The authors derive the following formula for $f(x)$:

$f(x) = [(1/8)x(x+2)(2x+1)]$ where [ ] denotes the greatest integer function.

Additional information concerning this problem can be found in the *Mathematical Gazette*. The problem appears in the article *How many triangles* by F. Gerrish (see *Mathematical Gazette* 54 (1970), pp. 241–246). Betty D. Mastrantone gives an elementary approach to the solution using triangular numbers (see *Mathematical Gazette* 55 (1971), pp. 438–440). Immediately following the comment by Mastrantone, B. W. Martin develops the solution

$$f(x) = \begin{cases} (1/8)x(2x+1)(x+2) & \text{if } x \text{ is even} \\ (1/8)(x+1)(2x^2+x-1) & \text{if } x \text{ is odd.} \end{cases}$$

2. *Solution by L. Carlitz and Richard Scoville, Duke University.*

Let the given triangle have a base of length n. Let $f_0(n)$ denote the number of triangles with apex above the base and $f_1(n)$ the number of triangles with apex below the base. The number of triangles of the first kind with apex on the kth line from the top is evidently $n-k$, so that

$$f_0(n) = \sum_{k=0}^{n-1} (k+1)(n-k) = \frac{1}{6}n(n+1)(n+2).$$

To evaluate $f_1(n)$, let s_k denote the number of triangles (of the second kind) with apex on the $(k+1)$st line. A glance at the diagram gives

$$s_1 = 1$$
$$s_2 = 1+1$$
$$s_3 = 1+2+1$$
$$s_4 = 1+2+2+1$$
$$s_5 = 1+2+3+2+1$$
$$s_6 = 1+2+3+3+2+1$$

and so on. Generally

$$\begin{cases} s_{2k} = s_{2k-1} + k \\ s_{2k+1} = s_{2k} + k + 1. \end{cases}$$

It follows that

$$s_{2k} = k(k+1), \; s_{2k-1} = k^2.$$

Thus

$$\begin{aligned} f_1^{(2m)} &= s_1 + s_2 + \cdots + s_{2m-1} \\ &= \sum_{k=1}^{m} s_{2k-1} + \sum_{k=1}^{m-1} s_{2k} \\ &= \sum_{k=1}^{m} k^2 + \sum_{k=1}^{m-1} k(k+1) \\ &= \tfrac{1}{6} m(m+1)(2m+1) + \tfrac{1}{3}(m+1)m(m-1) \\ &= \tfrac{1}{6} m(m+1)(4m-1), \end{aligned}$$

$$\begin{aligned} f_1(2m+1) &= f_1(2m) + s_{2m} \\ &= \tfrac{1}{6} m(m+1)(4m-1) + m(m+1) \\ &= \tfrac{1}{6} m(m+1)(4m+5). \end{aligned}$$

Since $f(n) = f_0(n) + f_1(n)$, we get

$$\begin{cases} f(2m) = \tfrac{1}{2} m(m+1)(4m+1) \\ f(2m+1) = \tfrac{1}{2}(m+1)(4m^2+7m+2). \end{cases}$$

Also solved by Gladwin Bartel, Otero Junior College, La Junta, Colorado; Donald Batman, M.I.T., Lincoln Laboratory, Lexington, Massachusetts; J. C. Binz, Bolligen, Switzerland; Albert A. Blank, Carnegie-Mellon University, Pittsburgh, Pennsylvania; Joseph C. Bodenrader, State University College of Arts and Science, Plattsburgh, New York; Michael Bolmer, Bucknell University; Arthur J. Bradley, Bendix Navigation & Control Division, Teterboro, New Jersey; Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania; Brother Alfred Brousseau. St. Mary's College, Moraga, California; Joseph B. Browne, Oklahoma State University; Scott Brown; Mary G. Checco, Pleasantville, New York; D. P. Choudhury, I. I. T., Kanpur, India; William D. Clewell, Baltimore, Maryland; Eliot

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Many solvers referred to the article by Charles L. Hamberg and Thomas M. Green, an application of *Triangular numbers to counting* which appeared in the Mathematics Teacher, Vol. LX, No 4, April 1967.

A Strange Game

891. [January 1974] *Proposed by John M. Howell, Littlerock, California.*

Teams A and B play a series of baseball games. Team A never swings the bat nor steals a base. Team B pitchers never hit a batter and throw strikes half the time. What is the average number of runs scored per game by Team A?

Solution by P. J. Pedler, The Flinders University of South Australia.

For each batter in team A,

$$P(\text{out}) = P(3 \text{ or more strikes in } 6 \text{ pitches})$$

$$= \frac{\binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}}{2^6}$$

$$= \frac{21}{32}$$

$$\Rightarrow P(\text{walk}) = 1 - P(\text{out}) = \frac{11}{32}.$$

In a given innings, the number of walks W before side A is out, is a negative binomial random variable with

$$P(W = w) = \binom{w+2}{w} \left(\frac{21}{32}\right)^3 \left(\frac{11}{32}\right)^w ; \; w = 0, 1, 2, \cdots,$$

and expected value $E(W) = \dfrac{3(11/32)}{21/32} = 1\frac{4}{7}.$

In a given innings, the number of runs R scored by side A is given by

$$R = \begin{cases} 0 & \text{if } W \leqq 3, \\ W-3 & \text{if } W > 3, \end{cases}$$

and hence $P(R = 0) = \sum_{w=0}^{3} P(W = w),$

$$P(R = r) = P(W = r + 3), \quad r > 0.$$

Now

$$\begin{aligned} E(R) &= \sum_{r=1}^{\infty} rP(R = r) \\ &= \sum_{w=4}^{\infty} (w-3)P(W = w) \\ &= E(W) - 3 + \sum_{w=0}^{3} (3-w)P(W = w). \end{aligned}$$

As

$$\sum_{w=0}^{3} (3-w)P(W = w)$$

$$= 3\binom{2}{0} \left(\frac{21}{32}\right)^3 \left(\frac{11}{32}\right)^0 + 2\binom{3}{1} \left(\frac{21}{32}\right)^3 \left(\frac{11}{32}\right)^1 + 1\binom{4}{2} \left(\frac{21}{32}\right)^3 \left(\frac{11}{32}\right)^2$$

$$= \frac{27366255}{16777216},$$

it follows that

$$E(R) = 1\frac{4}{7} - 3 + \frac{27366255}{16777216}$$

$$= \frac{23791625}{117440512}.$$

Hence the expected number of runs scored by team A in a nine innings game is

$$9E(R) = \frac{214124625}{117440512}$$

$$= 1.823260\ldots.$$

Also solved by Michael Bolmer, Bucknell University, Lewisburg, Pennsylvania; Robert X. Brennan, Dover, New Jersey; Eliot William Collins, New Paltz, New York; Milton P. Eisner, Michigan State University; Leon Gerber, St. John's University, Jamaica, New York; Michael Goldberg, Washington, D. C.; Ann Goodsell, University of Wisconsin-Oshkosh; Richard A. Groeneveld, Iowa State University, Ames, Iowa; John R. Heath, Minneapolis, Minnesota; Karl Heuer, Turnich, West Germany; John Howell, Howell Enterprises, Littlerock, California; J. C. Hudson, General Motors Institute, Flint, Michigan; Nancy E. Johnston, Immaculata College, Immaculata, Pennyslvania; Richard A. Jacobson, Houghton College, New York; Kay P. Litchfield, Provo, Utah; Joseph V. Michalowig; University of Santa Clara Problem Solving Seminar, Santa Clara, California; F.G. Schmitt, Jr., Berkeley, California; Thomas Spencer, Trenton State College, Trenton, New Jersey; R. S. Stacy, Manzano High School, Albuquerque, New Mexico, and the proposer.

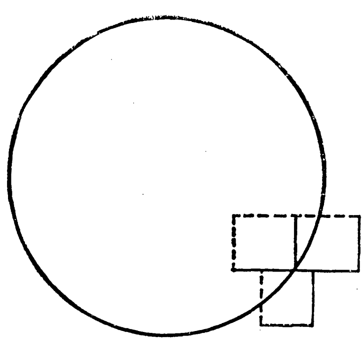
A Finite Set

892. [January 1974] *Proposed by Arnold Good, Lewis-St. Frances College, Illinois.*

Let S be a closed set in R^2, a circle will do; and let T be a disjoint collection of half-open intervals in R^2 covering S and satisfying:

1. Every interval of T contains at least one point of S.

2. Every point of S is interior to the union of, at most, four intervals of T (see figure).



Show that T contains only a finite number of intervals.

Solution by the proposer.

As each point of S is interior to, at most, four intervals of T, the union of these intervals, M_x, has x as an interior point. Hence, the collection $\{M_x^0 \mid x \in S\}$ is an open covering of S, and, thus, has a finite subcollection: $M_{x}^0, M_{x_2}^0, \cdots M_{x_n}^0$ covering S.

As each $M^0_{x_i}$, $i = 1, \cdots n$, is made up of no more than four intervals of T, a subcollection of T–call it T'–containing no more than $4n$ intervals, covers S.

We show that $T = T'$. Suppose it does not. Then at least one interval of T has been discarded. But this interval must contain at least one point of S, which—because T is a disjoint collection—can be in no other interval. Hence T' does not cover S. This is a contradiction and so $T = T'$, i.e., T is a finite collection.

Twin Primes

893. [January 1974] *Proposed by John Herschel, Mission Beach, California.*

Prove that the sum of twin primes whose sums is greater than or equal to 12 is always divisible by 12.

Solution by Ragnar Dybvik, Tingvoll, Norway.

Let the twin primes be $2n+1$ and $2n+3$, where n is an integer $\geqq 2$. If $n \equiv 1 \pmod 3$, $2n+1$ is divisible by 3, and so is $2n+3$, if $n \equiv 0 \pmod 3$. We then have $n \equiv 2 \pmod 3$ and can therefore set $2n+1 = 2(3N+2)+1 = 6N+5$ and $2n+3 = 6N+7$.

We then have $(2n+1)+(2n+3) = 12N+12 = 12(N+1)$, which obviously is divisible by 12.

Also solved by William B. Adams, Boston University, Boston, Massachusetts; Joe Albree, University of New Orleans; Leon Bankoff, Los Angeles, California; Merrill Barnebey, UW La Crosse; Gladwin E. Bartel, Otero Junnior College, La Junta, Colorado; Donald Batman, MIT, Lincoln Lab, Lexington, Massachusetts; Sister Marion Beiter, Rosary Hill College, Buffalo, New York; S. J. Benkoski, Virginia Beach, Virginia; A. Joseph Berlau, Hartsdale, New York; Martin Berman, Bronx Community College, Bronx, New York; Donald Braffitt, Armstrong State College, Savannah, Georgia; David C. Brooks, Seattle Pacific College, Seattle, Washington; Brother Alfred Brousseau, St. Mary's College, Moraga, California; D. P. Choudhury, I. I. T., Kanpur, India; Stephen C. Currier, Jr., Penn State University, Altoona, Pennsylvania; Marisa Diaz, University of Puerto Rico, Guagnabe, Puerto Rico; Stephen Dillard, Westmont College, Santa Barbara, California; Clayton W. Dodge, University of Maine at Orono; Roy Dubisch; Hugh M. Edgar, California State University, San Jose, California; Thomas E. Elsner; Alex G. Ferrer, Ensenada Baja, California; Marjorie A. Fitting, California State University, San Jose, California; Stanley Fox, City College of New York; Chuck Friesen, University of Nebraska, Lincoln, Nebraska; Richard A. Gibbs, Fort Lewis College, Durango, Colorado; Reinaldo E. Giudici, Universidad Simon Bolivar, Caracas, Venezuela; Donald A. Happel, Central Michigan University, Mt. Pleasant, Michigan; Heiko Harborth, TU Braunschweig, Germany; John Huschel, Mission Beach, California; Nancy J. Hazley, I. U. P., Indiana, Pennsylvania; Karl Heuer, Turnich, West Germany; Joseph C. Hudson, General Motors Institute, Flint, Michigan; J. A. H. Hunter, Toronto, Canada; Richard A. Jacobson, Houghton College, Houghton, New York; Paul T. Karch, University of Pittsburgh, Pennsylvania; John Herschel, Mission Beach, California; Margaret J. Kenney, Boston College, Chestnut Hill, Massachusetts; Lew Kowarski, Morgan State College, Baltimore, Maryland; Bob Prielipp, University of Wisconsin-Oshkosh; R. Leifer, Pittsburgh, Pennsylvania; Henry S. Lieberman, John Hancock Mutual Life Insurance, Boston, Massachusetts; Kay P. Litchfield, Provo, Utah; Graham Lord, Temple University, Philadelphia, Pennsylvania; Robert A. Meyer, University of Nebraska-Lincoln, Lincoln, Nebraska; James W. McHutchion, Ohio State University, Columbus, Ohio; George A. Novacky, Teaching Fellow, University of Pittsburgh; William Nuesslein; Albert J. Patsche, Rock Arsenal, Rock Island, Illinois; Sidney Penner, Bronx Community College, New York; Thomas L.

Reynolds, College of William and Mary in Virginia, Williamsburg, Virginia; Lawrence A. Ringenberg, Eastern Illinois University, Charleston, Illinois; Paul St. Laurent, Fall River, Massachusetts; Long Sang Chiu and Man Keung Siu (jointly), University of Miami, Coral Gables, Florida; Louis Sass, Colorado Mountain College, Leadville, Colorado; University of Santa Clara Problem Solving Seminar; Joseph Silverman, Brown University, Providence, Rhode Island; Paul Smith, University of Victoria; Roland Smith, Russell Sage College, Troy, New York; R. S. Stacy, Manzano High School, Albuquerque, New Mexico; Alan H. Stein, University of Connecticut, Waterbury, Connecticut; Ely Stern, New York City Community College, New York; Wesley T. Thompson, University of Arkansas; Charles W. Trigg, San Diego, California; Gillian W. Valk, Tucker, Georgia; Julius Vogel, Prudential Life Insurance, Newark, New Jersey; Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada; Harry Whitcomb, Philadelphia College, Philadelphia, Pennsylvania; Kenneth M. Wilke; Jim William, Michigan State University; Gerhard N. Wollan, Purdue University, Lafayette, Indiana; Sam Zaslavsky, Baruch College, New York; Charles Ziegenfus, Madison College, Harrisonburg, Virginia; Gene Zirkel, Nassau Community College; A. Zujus; and the proposer.

Comment on Problem 863

863. [March, 1973] *Proposed by K. W. Schmidt, University of Manitoba, Canada.*

The number of (-1)'s of an n-order Hadamard matrix is bounded by $n[n \pm (\sqrt{2n-1} - 1)]/2$.

Comment by Edward T. H. Wang, Wilfrid Laurier University, Canada.

Concerning problem 863 (this MAGAZINE, Vol. 46, March 1973, No. 2, p. 103) I would like to point out that the published "solution" by Prof. Goldberg (this MAGAZINE, Vol. 47, Jan. 1974, No. 1, p. 54) is definitely incorrect. It is by no means true that the "Hadamard matrix, in normal form, has the minimum number of (-1)'s, namely $n(n-1)/2$", as stated by Prof. Goldberg; e.g., the 4×4 Hadamard matrix has only 4 (-1)'s while $n(n-1)/2 = 6$. (Incidentally, Prof. Ryser should

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

not be held responsible for this mistake since this statement is definitely not in his monograph *Combinatorial Mathematics.*)

Furthermore, note that negating all the rows of the above matrix would yield a Hadamard matrix with 12 (-1)'s. Certainly $12 > n(n+1)/2 = 10$. In general, starting with any normalized Hadamard matrix of order $n \equiv 0 \pmod 4$, we can negate the 1st row and then the 1st column to obtain a Hadamard matrix with $2(n-1) + (n/2)(n-1) = (n-1)(n+4)/2$ (-1)'s. Clearly, $(n-1)(n+4)/2 > n(n+1)/2$, the bound quoted by Prof. Goldberg. Nevertheless, the Hadamard matrix obtained in this manner does *not* always give the maximum number of (-1)'s (though it does for $n = 4$); e.g., for $n = 16$, $(n-1)(n+4)/2 = 150$. However, if we consider the Kronecker product $H \otimes H$, where H is a 4×4 Hadamard matrix with 12 (-1)'s and 4 (-1)'s, then it is easily seen that $H \otimes H$ has $12^2 + 4^2 = 160$ (-1)'s. Negating all the rows of $H \otimes H$ then yields a 16×16 Hadamard matrix with 160 (-1)'s.

ANSWERS

A 603. If $f'(x) = f(-x)$, then $f''(x) = -f'(-x)$, where $f'(-x) = f(x)$. Hence $f''(x) = -f(x)$, or $f''(x) + f(x) = 0$. This second order linear differential equation has $f(x) = \sin x + \cos x$ as its solution.

A 604.

$$\begin{aligned} \sin 80^\circ &= 2\sin 40^\circ \cos 40^\circ \\ &= 4\sin 20^\circ \cos 20^\circ \cos 40^\circ \\ &= 8\sin 10^\circ \cos 10^\circ \cos 20^\circ \cos 40^\circ \end{aligned}$$

So $$\sin 80^\circ = 8\sin 10^\circ \sin 80^\circ \sin 70^\circ \sin 50^\circ.$$

Dividing both sides by $\sin 80^\circ$ and noting $\sin 30^\circ = \frac{1}{2}$, the desired result is obtained.

A 605. The result follows from $x^2 + y^2 = z^2$ and $x^2 + y^2 > 2xy$ since $x \neq y$.

A 606. More generally, it will follow by induction that

$$\{\Sigma x_i\}^{\Sigma x_i} \geqq \prod x_i^{x_i} \qquad (x_i > 0)$$

if we first show that

$$(x + y)^{x+y} \geqq x^x y^y.$$

Letting $y = kx$, we get the obvious inequality

$$(1 + k)(1 + k)^k \geqq k^k.$$

Another solution, but not as elementary, follows from the concavity of $\log x$:

$$\frac{\Sigma x_i \log x_i}{\Sigma x_i} \leqq \log \frac{\Sigma x_i^{\,2}}{\Sigma x_i} \leqq \log \ \Sigma x_i.$$

There is equality *iff* all the x_i's but one are zero.

A 607. The set of points given by $S = \{(\pi m, \pi n) \mid m \text{ and } n \text{ are rational}\}$ is easily proved to be dense in the plane. The distances between any two distinct points S are all of the form πr where r is algebraic. Since these distances are transcendental they are irrational. The set S, therefore, has the desired property.

(Quickies on page 287.)

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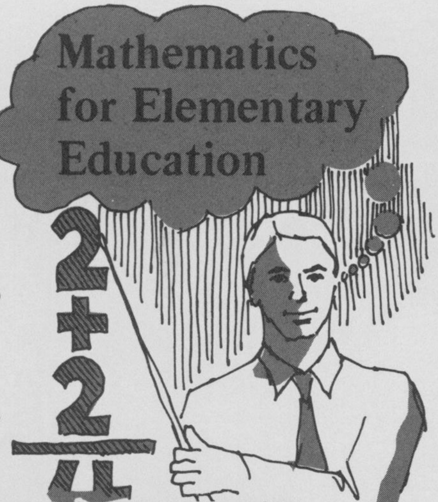

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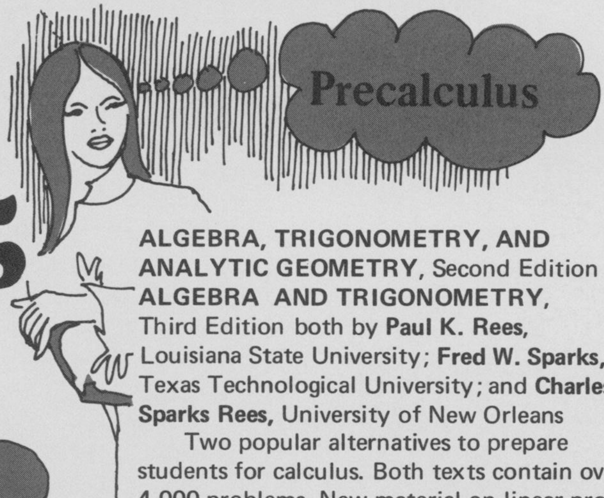

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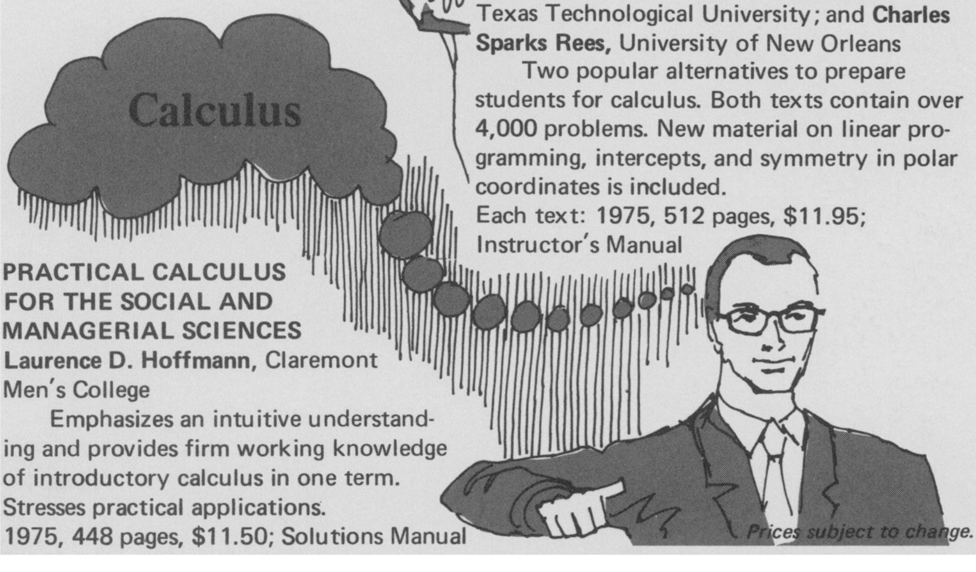

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